Idiosyncratic Volatility and the ICAPM Covariance Risk

Bing Han and Gang Li*

Abstract

We show theoretically and empirically that the cross-section of stock return idiosyncratic volatilities contains useful information about the ICAPM. We construct a proxy cross-sectional bivariate idiosyncratic volatility (CBIV) for the covariance risk between the market and the unobserved hedge portfolio under the ICAPM. Consistent with the ICAPM pricing relation, CBIV is a robust and significant predictor of the equity risk premium. We further show that the return predictability of the tail index in Kelly and Jiang (2014) can be explained by the ICAPM covariance risk.

JEL classification: G12, G13, G14, G17

Keywords: idiosyncratic volatility, intertemporal capital asset pricing model, covariance risk, tail risk, time-series stock market return predictability

---

* Han, binghan@cuhk.edu.hk and bing.han@rotman.utoronto.ca, University of Toronto Rotman School of Management and Chinese University of Hong Kong; Li (corresponding author), gang.li@cuhk.edu.hk, Chinese University of Hong Kong CUHK Business School. We appreciate value comments from Hendrik Bessembinder (editor), an anonymous referee, Turan Bali, Jay Cao, Kalok Chan, Darwin Choi, Alexandre Corhay, Dashan Huang, Yoontae Jeon, Wenxi Jiang, Raymond Kan, Charles Martineau, Tom McCurdy, Chayawat Ornthanalai, Dave Rapach, Mike Simutin, Jincheng Tong, William Waller, Yizhou Xiao, Nancy Xu, Yanhui Zhao, as well as seminar participants at Beijing Institute of Technology, Cheung Kong Graduate School of Business, Chinese University of Hong Kong (CUHK), CUHK Shenzhen, Dongbei University of Finance and Economics, London School of Economics and Political Science, Shanghai University of Finance and Economics, Southwestern University of Finance and Economics, Tianjin University, University of International Business and Economics, University of South Carolina, University of Sussex, University of Toronto, APAD (2020), NFA (2020), FMA (2020), Frontiers of Factor Investing (2021), ITAM (2021), AsiaFA (2021), and CIRF (2021) for very helpful comments. An earlier version of this paper is titled "Idiosyncratic Volatility and the Intertemporal Capital Asset Pricing Model."

I. Introduction

The seminal works by Merton (1973), Long (1974), Cox, Ingersoll, and Ross (1985), and Campbell (1993) build a solid theoretical background of the intertemporal capital asset pricing model (ICAPM). As an extension of the CAPM, the ICAPM involves additional factors corresponding to state variables that are related to time-varying investment opportunities. An important implication of the ICAPM is that the equity risk premium is determined by the variance of the market portfolio as well as the covariance between the market portfolio and an aggregate hedge portfolio (i.e., the ICAPM covariance risk).

A large literature has proposed different ways to estimate the market variance, but less work has been undertaken on the covariance risk. A key challenge in measuring the ICAPM covariance risk is that the hedge portfolio is unobservable. This paper proposes a new way to estimate the ICAPM covariance risk using the cross-section of stock idiosyncratic volatilities. We theoretically demonstrate that stock return idiosyncratic volatility under the market model (which is misspecified relative to the true ICAPM) contains useful information about the covariance between the market portfolio and the hedge portfolio. A proxy of the ICAPM covariance risk can be obtained from two differentially weighted average idiosyncratic volatilities. Such covariance proxy should be positively related to the equity risk premium, if the ICAPM pricing relation holds.

Empirically, we employ a dual-predictor system consisting of two differentially weighted average idiosyncratic volatilities (which we name IV_t^F and IV_t^S) to forecast aggregate stock returns. We find IV_t^F and IV_t^S jointly have strong and significant forecasting power for future stock market returns with an out-of-sample R^2 of 0.78% (resp. 8.85%) at three-month (resp. one-year) forecast horizon. In addition, the ratio of IV_t^F and IV_t^S, which we term the cross-sectional bivariate idiosyncratic volatility (CBIV), serves as a proxy of the ICAPM covariance risk and also has a strong predictive power for the stock market returns with an out-of-sample R^2 of 2.80% (13.68%) for three-month (one-year) forecast horizon. Our results are robust to variations in the constructions of covariance risk proxy. The return predictive power of our ICAPM covariance risk remains significant with little changes in magnitudes after controlling for alternative conditional covariance measures (e.g., Guo and Whitelaw (2006), Rossi and Timmermann (2015)) as well as existing predictors of stock market returns.

Our paper contributes to the estimation of the ICAPM covariance risk. Unlike previous studies, our approach is model free in the sense that it does not need to specify a proxy of the hedge portfolio or the state variables that drive the conditional covariance between the market portfolio and the hedge portfolio.[1] Our estimation of the covariance risk from the cross-section of stock idiosyncratic volatilities is motivated by the ICAPM theory and does not require any additional data beyond stock returns. Our approach can be applied at high frequencies such as daily and weekly. Moreover, results from both in-sample and out-of-sample tests show that compared to existing conditional covariance measures in the literature, our covariance risk proxy contains substantial new information about future stock market returns.

Our paper contributes to a large literature on stock market return predictability. Several recent studies document that stock market returns are predictable although most of the predictors either lose their significance in recent years or are only available for relatively short sample periods (Goyal, Welch, and Zafirov (2024)). In contrast, the stock return predictive power by our

---

[1] Guo and Whitelaw (2006) assume that the conditional covariance is a linear function of some observed state variables. Bali (2008) applies a bivariate GARCH model to estimate a portfolio's conditional covariance with the market. Bali and Engle (2010) apply the dynamic conditional correlation model of Engle (2002) to estimate the time-varying covariances between individual stock returns and market returns. Rossi and Timmermann (2015) propose a method for constructing the conditional covariance risk using a summary measure of economic activity to track time-varying investment opportunities.

covariance proxy CBIV is robust over a long sample period from 1960 to 2022 and holds for both short- and long-term forecast horizons. The significant market return predictability by CBIV is consistent with the ICAPM pricing relations. Formal specification tests using our covariance risk measure suggest that the ICAPM is correctly specified. Moreover, incorporating the covariance risk in forecasting stock market returns leads to robust evidence for a significantly positive relationship between the equity risk premium and the market variance as well as reasonable magnitudes of risk aversion estimates.

Our paper provides an alternative interpretation of the findings in Kelly and Jiang (2014). Motivated by the power law distribution, Kelly and Jiang (2014) construct a tail index using the cross-section of idiosyncratic stock returns. Their measure has a strong predictive power for stock market returns, which they interpret as evidence that tail risk matters for asset pricing. It remains one of the best time-series predictors in recent periods (Goyal et al. (2024)). We show theoretically that the Kelly and Jiang (2014) tail index is approximately proportional to the covariance risk under the ICAPM. Empirically, the Kelly and Jiang (2014) tail index and CBIV track each other closely with a correlation of 0.818. Both significantly forecast stock market returns in univariate predictive regressions, but the Kelly and Jiang (2014) tail index loses significance in the presence of CBIV in predictive regressions. Thus, the Kelly and Jiang (2014) index could manifest the ICAPM covariance risk rather than the tail risk. This sheds new light on the interpretation of the Kelly and Jiang (2014) tail index.

Idiosyncratic volatilities can matter for asset pricing due to market frictions, behavioral biases, or common factor in idiosyncratic volatilities, as pointed out by the literature. Our paper does not rely on any of these considerations. Our paper differs markedly from several recent studies showing that average or common idiosyncratic volatility is a priced risk factor (e.g., Chen and Petkova (2012), Herskovic, Kelly, Lustig, and Nieuwerburgh (2016)). We are agonistic about the source of priced risks captured by the unobserved hedge portfolio. Our insight is that the idiosyncratic volatilities serve as instruments for the covariance risk under the ICAPM. We show that the cross-section of idiosyncratic volatility can be used to estimate the covariance risk between the market and the unobserved hedge portfolio. This link and the ICAPM pricing relation underly the ability of our novel measures based on stock idiosyncratic volatilities to forecast stock market returns.

The remainder of this paper is organized as follows. Section II derives theoretically the relationship between idiosyncratic volatility and the conditional covariance risk under the ICAPM. Section III presents robust evidence on the stock return predictive power by our covariance risk proxy derived from the cross-section of idiosyncratic volatilities. Our results hold for both in-sample and out-of-sample tests. Section IV conducts ICAPM specification tests. Section V examines the relationship between the covariance risk under the ICAPM and the tail index of Kelly and Jiang (2014). Section VI concludes the paper. All proofs are provided in the appendix.

II. Theoretical Framework

We start with the ICAPM under which the conditional expected return of a risky security i is given by (see, e.g., Merton (1973), Long (1974), Cox et al. (1985), and Campbell (1993)):

$$(1) \qquad E_t(R_{i,t+1}) = \gamma_M Cov_t(R_{i,t+1}, R_{M,t+1}) + \sum_{k=1}^{K} \gamma_k Cov_t(R_{i,t+1}, \Delta Z_{k,t+1}), \qquad i = 1, \ldots, N,$$

where $R_{i,t+1}$ is the excess return (rates of return minus a risk-free rate) for security i, $R_{M,t+1}$ is the market excess return, and the $Z_k{}'$s are relevant state variables that contain information about future investment opportunities. γ_M is the relative risk aversion of the representative agent. γ_k is the weighted average across investors of their state-variable aversions. E_t and Cov_t denote the conditional expectation and the conditional covariance based on information at time *t.* Equation (1) gives the following conditional equity risk premium:

$$(2) \qquad E_t(R_{M,t+1}) = \gamma_M Var_t(R_{M,t+1}) + \sum_{k=1}^{K} \gamma_k Cov_t(R_{M,t+1}, \Delta Z_{k,t+1}).$$

Unlike the one-factor CAPM model, the ICAPM states that the conditional equity risk premium is determined not only by the conditional variance of the market portfolio but also by the conditional covariance of the market portfolio with innovations of the relevant state variables. By utilizing the concept of factor mimicking portfolio or hedge portfolio "H" of traded securities whose return is maximally correlated with the innovation of the state variables, equations (1) and (2) can be simplified to give the following expressions for the risk premia under the ICAPM:

$$(3) \qquad \mu_{i,t} = \gamma_M \sigma_{iM,t} + \gamma_H \sigma_{iH,t}.$$

$$(4) \qquad \mu_{M,t} = \gamma_M \sigma_{M,t}^2 + \gamma_H \sigma_{MH,t}.$$

$$(5) \qquad \mu_{H,t} = \gamma_M \sigma_{MH,t} + \gamma_H \sigma_{H,t}^2.$$

Here and in the rest of the paper, we use μ to denote expected returns and σ^2 to denote variance-covariance terms. Specifically, $\mu_{i,t}, \mu_{M,t}, \mu_{H,t}$ $(\sigma_{i,t}, \sigma_{M,t}, \sigma_{H,t})$ are the conditional expected excess returns (volatilities) of security i, the market portfolio, and the hedge portfolio at time t. $\sigma_{iM,t}, \sigma_{iH,t}, \sigma_{MH,t}$ represent the corresponding covariance terms. Following the literature (e.g., Bali (2008), Bali and Engle (2010), and Rossi and Timmermann (2015)), we refer to the covariance term $\sigma_{MH,t}$ as the ICAPM covariance risk.

Based on Ingersoll ((1987), p.218), a beta representation of equation (3) can be written as:

$$\mu_{i,t} = \beta_{iM,t}\mu_{M,t} + \beta_{iH,t}\mu_{H,t}, \quad (6)$$

where $\beta_{iM,t}$ and $\beta_{iH,t}$ are the exposures of security i to the market and hedge portfolios given by the following regression model:

$$R_{i,t+1} = \beta_{iM,t}R_{M,t+1} + \beta_{iH,t}R_{H,t+1} + \varepsilon_{i,t+1}, \quad (7)$$

where $R_{M,t+1}$ and $R_{H,t+1}$ are the excess returns of the market portfolio and the hedge portfolio, and $\varepsilon_{i,t+1}$ is the true idiosyncratic return of security i. Despite the importance of the ICAPM covariance risk $\sigma_{MH,t}$, it is difficult to estimate since the hedge portfolio H is unobserved. We propose a new approach to estimate $\sigma_{MH,t}$ that differs markedly from previous studies that model it as a function of some observed state variables (e.g., Rossi and Timmermann (2015)). Our approach relies on the ICAPM pricing relationships. The insight is that, under the ICAPM, the idiosyncratic variance relative to the misspecified one-factor market model contains useful information about the covariance risk.

Assume that econometricians only use the single-index model:

$$R_{i,t+1} = b_{iM,t}R_{M,t+1} + \eta_{i,t+1}, \quad (8)$$

where $b_{iM,t} = \frac{\sigma_{iM,t}}{\sigma^2_{M,t}}$ is the market beta, and $\eta_{i,t+1}$ is the (misspecified) idiosyncratic return of security *i* under the one-factor market model. Proposition 1 demonstrates that both the first and the second conditional moments of the idiosyncratic return $\eta_{i,t+1}$ can be expressed as a combination of the conditional variance of the hedge portfolio and the conditional covariance between the market portfolio and the ICAPM hedge portfolio.

Proposition 1. *The time-t conditional mean and variance of the misspecified firm idiosyncratic return $\eta_{i,t+1}$ obtained from the single-index model in equation (8) are related to the conditional variance $\sigma^2_{H,t}$ of the hedge portfolio H and the conditional covariance $\sigma_{MH,t}$ under the ICAPM by:*

$$E_t(\eta_{i,t+1}) = \gamma_H b_{iH,t}\sigma^2_{H,t} - \gamma_H b_{iM,t}\sigma_{MH,t}, \tag{9}$$

$$Var_t(\eta_{i,t+1}) = \beta_{iH,t} b_{iH,t}\sigma^2_{H,t} - \beta_{iH,t} b_{iM,t}\sigma_{MH,t} + \sigma^2_{\varepsilon_i,t}, \tag{10}$$

where $\begin{cases} \beta_{iM,t} = \frac{\sigma_{iM,t}\sigma^2_{H,t} - \sigma_{iH,t}\sigma_{MH,t}}{\sigma^2_{H,t}\sigma^2_{M,t} - \sigma^2_{MH,t}} \\ \beta_{iH,t} = \frac{\sigma_{iH,t}\sigma^2_{M,t} - \sigma_{iM,t}\sigma_{MH,t}}{\sigma^2_{H,t}\sigma^2_{M,t} - \sigma^2_{MH,t}} \end{cases}$ *and* $\begin{cases} b_{iM,t} = \frac{\sigma_{iM,t}}{\sigma^2_{M,t}} \\ b_{iH,t} = \frac{\sigma_{iH,t}}{\sigma^2_{H,t}} \end{cases}$.

Proposition 1 reveals that the idiosyncratic variance under the market model contains useful information about $\sigma_{MH,t}$ which is a key determinant of the equity risk premium under the ICAPM as shown by equation (4). This argument for why idiosyncratic variance matters for asset pricing is different from related studies in the literature.

Proposition 1 suggests that individual stock idiosyncratic variance could be useful for forecasting stock market returns via its correlation with $\sigma_{MH,t}$, but it is likely a noisy predictor because idiosyncratic variance also depends on the variance $\sigma^2_{H,t}$ of the hedge portfolio which is irrelevant for predicting stock market returns. Pooling idiosyncratic variances across stocks to estimate $\sigma_{MH,t}$ can help improve the information-noise ratio. For example, one can attempt to extract the relevant information (i.e., about $\sigma_{MH,t}$) from the individual stock idiosyncratic variances using principal component analysis or the three-pass regression filter as in Kelly and Pruitt (2015). Below, we propose a simple alternative method to estimate the conditional covariance $\sigma_{MH,t}$ using two differentially weighted average idiosyncratic variances.

Corollary 1.1. *Under the ICAPM, the conditional variance of the hedge portfolio H and the conditional covariance between the market portfolio M and the hedge portfolio H can be expressed in terms of two cross-sectional average idiosyncratic variances IV_t^F and IV_t^S with different weights $w_{i,t}^F$ and $w_{i,t}^S$:*

$$\sigma_{H,t}^2 = \frac{B_t^S}{A_t^F B_t^S - A_t^S B_t^F}\widetilde{IV}_t^F - \frac{B_t^F}{A_t^F B_t^S - A_t^S B_t^F}\widetilde{IV}_t^S, \tag{11}$$

$$\sigma_{MH,t} = \frac{A_t^S}{A_t^F B_t^S - A_t^S B_t^F}\widetilde{IV}_t^F - \frac{A_t^F}{A_t^F B_t^S - A_t^S B_t^F}\widetilde{IV}_t^S, \tag{12}$$

where $\begin{cases} A_t^F = \sum_{i=1}^{N_t} w_{i,t}^F \beta_{iH,t} b_{iH,t} \\ A_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \beta_{iH,t} b_{iH,t} \end{cases}, \begin{cases} B_t^F = \sum_{i=1}^{N_t} w_{i,t}^F \beta_{iH,t} b_{iM,t} \\ B_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \beta_{iH,t} b_{iM,t} \end{cases}, \begin{cases} IV_t^F = \sum_{i=1}^{N_t} w_{i,t}^F Var_t(\eta_{i,t+1}) \\ IV_t^S = \sum_{i=1}^{N_t} w_{i,t}^S Var_t(\eta_{i,t+1}) \end{cases},$

$\begin{cases} \Omega_t^F = \sum_{i=1}^{N_t} w_{i,t}^F \sigma_{\varepsilon_i,t}^2 \\ \Omega_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \sigma_{\varepsilon_i,t}^2 \end{cases}, \quad and \quad \begin{cases} \widetilde{IV}_t^F = IV_t^F - \Omega_t^F \\ \widetilde{IV}_t^S = IV_t^S - \Omega_t^S \end{cases}.$

Theoretically, $\sigma_{MH,t}$ is a combination of $\widetilde{IV}_t^F$ and $\widetilde{IV}_t^S$ instead of IV_t^F and IV_t^S. The difference between $\widetilde{IV}_t^F$ and IV_t^F (or $\widetilde{IV}_t^S$ and IV_t^S) is given by weighted average of the true idiosyncratic variances. Since the true idiosyncratic variances are not related to the equity risk premium, we can effectively use IV_t^F and IV_t^S when predicting stock market returns. Proposition

1 equation (9) suggests that $E_t(\eta_{i,t+1})$, the mean of the idiosyncratic return $\eta_{i,t+1}$ under the CAPM, also contains useful information about the ICAPM covariance risk $\sigma_{MH,t}$. We explore this in Section V when linking $\sigma_{MH,t}$ to the Kelly and Jiang (2014) tail index.

The next proposition follows from the ICAPM pricing relationships in equations (4)-(5) together with Corollary 1.1.

Proposition 2. *Under the ICAPM, the conditional equity risk premium can be expressed as:*

$$(13) \qquad \mu_{M,t} = \gamma_M \times \sigma_{M,t}^2 + C_t^F \times \widetilde{IV}_t^F - C_t^S \times \widetilde{IV}_t^S \approx \gamma_M \sigma_{M,t}^2 + \psi_0 \ln\left(\frac{C_t^F}{C_t^S}\right) + \psi_0 \ln\left(\frac{\widetilde{IV}_t^F}{\widetilde{IV}_t^S}\right),$$

where $C_t^F = \frac{\gamma_H A_t^S}{A_t^F B_t^S - A_t^S B_t^F}$, $C_t^S = \frac{\gamma_H A_t^F}{A_t^F B_t^S - A_t^S B_t^F}$, *and* $\psi_0 = \frac{E(C_t^F IV_t^F) + E(C_t^S IV_t^S)}{2}$.

Proposition 2 highlights that under the ICAPM, two average idiosyncratic variances with different weights can be used to predict stock market returns. This dual-predictor system can be further simplified to a single predictor which is the log-ratio of two average idiosyncratic variances.[2] Equation (13) is equivalent to the following relation:

$$(14) \qquad \sigma_{MH,t} \approx \frac{1}{\gamma_H}\left[\psi_0 \ln\left(\frac{C_t^F}{C_t^S}\right) + \psi_0 \ln\left(\frac{\widetilde{IV}_t^F}{\widetilde{IV}_t^S}\right)\right].$$

Equation (14) suggests that the logarithm of the ratio of two different weighted average idiosyncratic variances can proxy for the covariance risk under the ICAPM. Given logarithm is a monotonically increasing function, in the empirical analyses we simply use the ratio of two

---

[2] In deriving Proposition 2, we apply the technique of log approximation which has been used by Campbell and Shiller (1988) and other studies.

different weighted average idiosyncratic variances as a proxy for $\sigma_{MH,t}$. We name this proxy of the covariance risk the "cross-sectional bivariate idiosyncratic volatility" (CBIV):

$$CBIV_t \equiv \frac{\widetilde{IV}_t^F}{\widetilde{IV}_t^S}. \tag{15}$$

In Section III, we show that CBIV captures the same information as the dual-predictor system of idiosyncratic volatilities when predicting stock market returns.

III. ICAPM Covariance and the Equity Risk Premium

A. New Proxies of the ICAPM Covariance Risk

We construct proxies of the ICAPM covariance risk using the cross-section of stock idiosyncratic volatilities (i.e., CBIV). Then we test the ability of our covariance risk measure to forecast stock market returns, as suggested by the ICAPM pricing relation. The testing period is from 1960 to 2022. To make our empirical results comparable to related papers in the literature, we use idiosyncratic volatility instead of idiosyncratic variance. We compute idiosyncratic volatility as the standard deviation of residuals from the one-factor market model. For each stock in each month, we use daily returns over the past 60 trading days (we require at least 20 return observations) to fit the market model. Our empirical results do not change materially if idiosyncratic volatility based on different benchmarks (e.g., Fama-French three- or five-factor model), estimation window based on different trading days, or idiosyncratic variance of stock returns are used. We obtain from the Center for Research in Security Prices (CRSP) data on stock returns, stock prices, and shares outstanding. We download the stock market excess returns (MKTRF) and the Fama-French factors from Kenneth French's website.

We have shown theoretically that two different idiosyncratic volatilities can be used to estimate the ICAPM covariance risk $\sigma_{MH,t}$ (Corollary 1.1) and forecast stock market returns (Proposition 2). To ensure the robustness of our empirical results, we try all combinations of two percentiles (from the 1st to the 99th percentile) of the cross-section of individual stock idiosyncratic volatilities. When defining each percentile of idiosyncratic volatilities, we first subtract the cross-sectional mean of idiosyncratic volatility in order to isolate the potential effect of the time-series trend in the aggregate idiosyncratic volatility (e.g., Campbell, Lettau, Malkiel, and Xu (2001), Bekaert, Hodrick, and Zhang (2012)):

$$IV_{n,t} = Q_t^C\big(Var_t(\eta_{i,t+1}), n\text{-}th\big) - E_t^C\left(Var_t(\eta_{i,t+1})\right), \tag{16}$$

where $Q_t^C\big(Var_t(\eta_{i,t+1}), n\text{-}th\big)$ is the $n\text{-}th$ quantile of the *cross-sectional* distribution of individual stock idiosyncratic volatility in month t, and $E_t^C\left(Var_t(\eta_{i,t+1})\right)$ is the equal-weighted average idiosyncratic volatility across all stocks. Since all individual idiosyncratic volatilities satisfy equation (10) in Proposition 1, the demeaned $IV_{n,t}$ above still contains useful information about the covariance risk $\sigma_{MH,t}$. We then use the ratio of a pair of $IV_{n,t}$ as a proxy for $\sigma_{MH,t}$:

$$CBIV_t = \frac{IV_{n,t}}{IV_{m,t}}, \tag{17}$$

where $(n, m) \in \{1, 2, 3, \ldots, 99\}$ and $n > m$. Across all 4,851 such combinations, we find consistent and robust evidence of significant stock market return predictability by either two predictors $IV_{n,t}$ and $IV_{m,t}$ given in equation (16) or by a single predictor $CBIV_t$ (see Section III.D).

For our main reported results, we pick two representative idiosyncratic volatilities IV_t^F and IV_t^S as follows. In each month t, and for each pair of (n, m), we run a univariate predictive regression of monthly stock market returns on the lagged value of the corresponding CBIV given in equation (17) using an expanding window of the data:[3]

$$(18) \qquad R_{M,s} = a_{n,m} + b_{n,m} \times CBIV_{s-1} + \epsilon_{M,s} = a_{n,m} + b_{n,m} \times \frac{IV_{n,s-1}}{IV_{m,s-1}} + \epsilon_{M,s}, \qquad s = 2, \ldots, t.$$

To ensure all CBIVs forecast stock market returns with a positive sign, we take the negative value of $IV_{m,t}$ if $b_{n,m} < 0$:

$$(19) \qquad IV_{m,t}^* = \frac{|b_{n,m}|}{b_{n,m}} \times IV_{m,t}.$$

We define IV_t^F and IV_t^S respectively as the median of $IV_{n,t}$'s and $IV_{m,t}^*$'s across all pairs of $(n, m) \in \{1, 2, 3, \ldots, 99\}$ and $n > m$:

$$(20) \qquad \begin{cases} IV_t^F = Q_t^C(IV_{n,t}, 50\text{-}th) \\ IV_t^S = Q_t^C(IV_{m,t}^*, 50\text{-}th) \\ CBIV_t = \dfrac{IV_t^F}{IV_t^S} \end{cases}.$$

Taking the median can reduce the effect of extreme values from the data and generate more robust predictors than taking the mean, especially given the distribution of volatility is highly positively skewed.[4]

---

[3] To ensure we have enough observations to run the regressions at the beginning of our testing period (i.e., 1960), we start the expanding window from 1950. Our empirical results are robust to alternative choices of starting point for the expanding window.
[4] Our empirical results remain significant if we take the cross-sectional mean, though the R^2's and *t*-statistics are somewhat smaller than the reported results using median in equation (20).

Figure 1 plots the monthly time series of ten quantiles of the cross-section of idiosyncratic volatilities, as well as two selected idiosyncratic volatilities IV_t^F, IV_t^S, and the corresponding CBIV. To compare CBIV with alternative measures of conditional covariance documented in the literature, we replicate two conditional covariance measures proposed by Guo and Whitelaw (2006) and Rossi and Timmermann (2015). We also compute the Kelly and Jiang (2014) tail index (KJ tail index) and collect the aggregate disagreement index (HLW Disp) by Huang, Li, and Wang (2021). Table 1 provides their summary statistics.

[Insert Figure 1 and Table 1 approximately here]

Table 1 Panel B reports the correlations among key variables. CBIV has a correlation of 0.282 with the covariance measure of Guo and Whitelaw (2006) and a correlation of 0.215 with the covariance measure of Rossi and Timmermann (2015). Huang et al. (2021) propose an aggregate disagreement measure based on multiple volume- and volatility-based variables, one of which is aggregate idiosyncratic volatility. However, CBIV and HLW Disp are largely independent with a correlation of -0.056, suggesting that they do not capture similar information. Furthermore, CBIV has a correlation of only -0.042 with the realized stock market volatility (SMV) based on the past 30-day daily returns of MKTRF. On the contrary, CBIV tracks the KJ tail index with a correlation of 0.818. In Section V, we will conduct a comprehensive empirical analysis to show that KJ tail index captures largely overlapping information as CBIV.

B. In-sample Return Predictability

Motivated by Proposition 2 and following the literature (e.g., Bali and Engle (2010), Rossi and Timmermann (2015), Rapach et al. (2016), and Huang et al. (2021)), we run the multi-period predictive regressions below:

$$
(21) \qquad \begin{cases} \sum_{k=1}^{K} \frac{r_{t+k}}{K} \equiv r_{t,t+K} = a + b_1 \times IV_t^F + b_2 \times IV_t^S + \epsilon_{t,t+K} \\ \sum_{k=1}^{K} \frac{r_{t+k}}{K} \equiv r_{t,t+K} = a + b \times CBIV_t + \epsilon_{t,t+K} \end{cases},
$$

where r_{t+k} is the value-weighted market excess return (MKTRF) at time $t + k$, IV_t^F and IV_t^S are two differentially weighted average idiosyncratic volatilities, and K represents the forecast horizon (in number of months). When $K > 1$, we adjust for serial correlation and conditional heteroskedasticity using the Newey-West correction with $K - 1$ lags (Newey and West (1987)). To make the coefficients comparable, we scale all independent variables to have zero mean and one standard deviation. In addition to using two idiosyncratic volatilities as regressors, we also run the univariate predictive regression using CBIV defined in equation (20). The results are reported in Table 2.

[Insert Table 2 approximately here]

Table 2 Panel A provides convincing evidence that IV_t^F and IV_t^S together significantly predict stock market returns. The in-sample three-month and one-year adjusted R^2's are 2.63% and 12.88% respectively. The joint predictive power of IV_t^F and IV_t^S are not driven by multicollinearity. In the bivariate regressions, the estimated coefficients of IV_t^F and IV_t^S are both statistically significant with small standard errors while in the univariate regressions, neither coefficient is significantly different from zero. This is opposite to the pattern of multicollinearity.

Table 2 Panel B provides consistent evidence that CBIV by itself is also a significant predictor of future stock market returns, especially over longer horizons. For example, the in-sample univariate regression with three-month (one-year) forecast horizon has an adjusted R^2 of 2.78% (11.80%) and a *t*-statistic of 3.40 (3.53) for the estimated CBIV coefficient. The explanatory power of the univariate regression using CBIV is comparable in magnitude to that of the corresponding bivariate regression using two idiosyncratic volatilities IV_t^F and IV_t^S.

In contrast to the strong and robust stock market return predictability by IV_t^F and IV_t^S jointly, neither variable alone can significantly predict stock market returns over our sample period from 1960 to 2022. This is consistent with findings in the previous studies that a single average idiosyncratic volatility does not reliably forecast stock market returns. For example, Goyal and Santa-Clara (2003) find that equal-weighted idiosyncratic volatility (EWIV) can significantly forecast future stock market returns, but Bali, Cakici, Yan, and Zhang (2005) and Wei and Zhang (2005) show that the predictive power of EWIV does not hold in an extended sample period. Several studies document that value-weighted idiosyncratic volatility (VWIV) is negatively related to future aggregate stock returns, although the predictive power is marginal when used alone.[5]

To confirm that our results are novel and robust, we run the following multiple predictive regressions of stock market returns on two different weighted average idiosyncratic volatilities or our covariance risk proxy CBIV, controlling for existing predictors ($X_{j,t}$) of the stock market returns:

[5] The relevant studies include Guo and Savickas (2008), Pollet and Wilson (2010), Chen and Petkova (2012), and Huang et al. (2021).

$$(22) \qquad \begin{cases} r_{t,t+K} = a + b_1 \times IV_t^F + b_2 \times IV_t^S + \sum_{j=1}^{M} c_j \times X_{j,t} + \epsilon_{t,t+K} \\ r_{t,t+K} = a + b \times CBIV_t + \sum_{j=1}^{M} c_j \times X_{j,t} + \epsilon_{t,t+K} \end{cases}.$$

The control variables include two alternative conditional covariance measures proposed by Guo and Whitelaw (2006) and Rossi and Timmermann (2015), the first three principal components of 14 predictors in Goyal and Welch (2008), aggregate stock illiquidity following Amihud (2002) and Chen, Eaton, and Paye (2018), and the average stock correlation according to Pollet and Wilson (2010).[6] We also collect data on investor sentiment (PLS Sentiment) by Huang, Jiang, Tu, and Zhou (2015) and aggregate disagreement index (HLW Disp) by Huang et al. (2021) from the corresponding authors' websites as control variables.[7]

[Insert Table 3 approximately here]

Table 3 shows that after controlling for these existing predictors, both IV_t^F and IV_t^S retain significance in predicting stock market returns. The signs and magnitudes of the coefficients of IV_t^F and IV_t^S in Table 3 are about the same as the corresponding ones in Table 2. We also run similar multiple predictive regressions using CBIV instead of both IV_t^F and IV_t^S but the same set of controls. Table 3 shows that CBIV remains a significant predictor of stock market returns in

[6] We follow Rapach, Ringgenberg, and Zhou (2016) to construct the three principal components. The raw data is collected from Amit Goyal's website.

[7] We consider other predictors of stock market returns such as investor sentiment by Baker and Wurgler (2007), market variance risk premium by Bollerslev, Tauchen, and Zhou (2009), market risk-neutral volatility index by Martin (2017), and aggregate implied volatility spread by Han and Li (2021). Since those predictors have shorter available sample periods, they are not included in our reported empirical results. In untabulated results, we confirm that our findings are robust to controlling for those predictors.

the presence of existing stock market return predictors. In particular, our new covariance risk measure CBIV remains a robust predictor of stock market returns after controlling for the two alternative measures of conditional covariance. We find that the conditional covariance measure in Rossi and Timmermann (2015) is a strong and significant predictor of the equity risk premium (beyond their sample period), while the conditional covariance based on Guo and Whitelaw (2006) does not significantly forecast stock market returns. These findings are consistent with Rossi and Timmermann (2015).

C. Out-of-sample Forecast Performance

We conduct out-of-sample tests by splitting the data sample (1960 to 2022) into two parts: 1960 to 1979 as the in-sample estimation period and 1980 to 2022 as the out-of-sample performance evaluation period. Starting in January 1980, we recursively run various predictive regressions each month using historical data from January 1960 and then compare the out-of-sample forecast errors (i.e., differences between the realized market returns and the predicted returns) with those from the benchmark model that uses the historical average as the forecast. Similar out-of-sample tests are used by Campbell and Thompson (2008), Goyal and Welch (2008), and Rapach, Strauss, and Zhou (2010). We adopt the Clark and West (2007) test of equal accuracy between the predictive regression in equation (23) and the unconditional benchmark model in equation (24) which can be nested as a special case of equation (23) that includes only an intercept in the regression:

$$\begin{cases} r_{t,t+K} = \alpha + \beta \times x_t + \epsilon_{t,t+K}, & t = 1, \dots, T_0 - K. \\ \hat{r}_{t,t+K} = \hat{\alpha} + \hat{\beta} \times x_t, & t = T_0, \dots, T. \end{cases} \tag{23}$$

$$(24) \qquad \textit{Benchmark:} \quad r^B_{t,t+K} = \frac{1}{t-K}\sum_{s=1}^{t-K} r_{s,s+K}\,, \quad t = T_0, \ldots, T,$$

where K is the forecast horizon, $r_{t,t+K}$ is the stock market excess return from time t to $t+K$, x_t is the value of the predictor at time t, and $\hat{r}_{t,t+K}$ is the forecasted return based on x_t from the recursive regression. The out-of-sample R^2 statistic is defined as one minus the ratio of the mean squared forecast error of the larger model to that of the benchmark model:

$$(25) \qquad R^2_{OS} = 1 - \left(\frac{MSFE_1}{MSFE_0}\right),$$

where $T - T_0$ is the number of out-of-sample evaluation periods, $MSFE_1 = \frac{1}{T-T_0}\sum_{t=T_0}^{T}\left(r_{t,t+K} - \hat{r}_{t,t+K}\right)^2$ and $MSFE_0 = \frac{1}{T-T_0}\sum_{t=T_0}^{T}\left(r_{t,t+K} - r^B_{t,t+K}\right)^2$. We test the hypothesis H_0: $MSFE_0 \leq MSFE_1$ $vs.$ H_1: $MSFE_0 > MSFE_1$, or equivalently H_0: $R^2_{OS} \leq 0$ $vs.$ H_1: $R^2_{OS} > 0$. Following the Clark and West (2007) test for nested models, we adjust the point estimate of the difference between two MSFEs for the noise associated with the larger model's forecast and define:

$$(26) \qquad \hat{f}_{t,t+K} = \left(r_{t,t+K} - r^B_{t,t+K}\right)^2 - \left[\left(r_{t,t+K} - \hat{r}_{t,t+K}\right)^2 - \left(r^B_{t,t+K} - \hat{r}_{t,t+K}\right)^2\right].$$

The test of equal predictive accuracy is conducted by regressing $\hat{f}_{t,t+K}$ on a constant and using the resulting *z*-statistic for a zero coefficient. The null hypothesis is rejected (i.e., equivalent to R^2_{OS} as statistically significant) if this statistic is greater than 1.282 for a one-sided test at 10% confidence, 1.645 for a one-sided test at 5% confidence, or 2.334 for a one-sided test at 1% confidence. When the forecast horizon K is greater than one, we adjust for serial correlation and conditional heteroskedasticity using the Newey and West (1987) correction with $K-1$ lags.

[Insert Table 4 approximately here]

Table 4 Panel A reports the R^2_{OS} statistics for various predictors and forecast horizons. The out-of-sample R^2_{OS} for the combination of IV^F_t and IV^S_t (resp. CBIV) is as high as 0.78% (resp. 2.80%) for three months ahead, 5.48% (resp. 7.46%) for six months ahead, and 8.85% (resp. 13.68%) for a one-year ahead forecast horizon. All of them are statistically significant at the 1% level. Interestingly, the single predictor CBIV consistently performs better in out-of-sample tests than the dual predictor system IV^F_t and IV^S_t.

The time-series predictability of stock market returns has important implications for market timing by guiding investors to optimally allocate wealth between stock investments and a risk-free asset (e.g., Kandel and Stambaugh (1996), Campbell and Thompson (2008), and Rapach et al. (2010)). Following the literature, we consider a mean-variance-utility investor who allocates wealth between the market portfolio and T-bills. Given an investment horizon of K periods, her optimal weight on the market portfolio is:

$$w_{t,t+K} = \frac{1}{\gamma}\frac{\hat{r}_{t,t+K}}{\hat{\sigma}^2_{t,t+K}}, \quad (27)$$

where $\hat{r}_{t,t+K}$ is the conditional expected market excess return (i.e., forecast based on a predictor) given by equation (23), $\hat{\sigma}^2_{t,t+K}$ is estimated from the historical monthly returns over the past five years, and the relative risk aversion γ is set to three. The portfolio is rebalanced every month. The corresponding Sharpe ratio of the investor's optimal portfolio is given by:

$$SR = \frac{\bar{R}_P}{\sigma_p}, \quad (28)$$

where $\bar{R}_P$ and σ_p are the mean and the standard deviation of the portfolio return. The average utility gain or the certainty equivalent return (CER) is computed as:

$$CER = \bar{R}_P - 0.5\gamma\sigma_p^2. \tag{29}$$

To gauge the economic benefit of a predictor to the mean-variance investor, we compare the CER above associated with the optimal portfolio based on the forecasts provided by the predictor to $\overline{CER}$, the certainty equivalent return of a benchmark portfolio formed based on the average return and the standard deviation estimated from historical returns. The difference is defined as the CER gain:

$$CER\ Gain = CER - \overline{CER}. \tag{30}$$

Table 4 Panel B reports the economic value of using out-of-sample forecasts by either IV_t^F or IV_t^S individually, as well as the economic value of using the two predictors together or CBIV to form the optimal portfolio. Consistent with the results in Table 4 Panel A, using the forecasts of the stock market returns provided by IV_t^F and IV_t^S together or by CBIV alone leads to large and positive certainty equivalent gain at all horizons from one month to two years. In contrast, there is no apparent certainty equivalent gain to using the forecast based on only one average idiosyncratic variance as predictor.

As an additional check of the out-of-sample forecast performance of our covariance risk measure CBIV, we adopt the forecast encompassing test to evaluate the incremental predictive power by CBIV relative to the alternative conditional covariance measures in Guo and Whitelaw (2006) and Rossi and Timmermann (2015). Motivated by Fair and Shiller (1989) and Harvey, Leybourne, and Newbold (1998), the idea is to form an optimal forecast $\hat{r}_{t,t+K}^*$ by combining the predicted value based on CBIV and that based on an existing predictor i:

$$\hat{r}_{t,t+K}^* = (1-\lambda)\hat{r}_{t,t+K}^i + \lambda\hat{r}_{t,t+K}^{CBIV}, \ \ 0 \le \lambda \le 1. \tag{31}$$

We want to find a λ so that the corresponding combined forecast in equation (31) is unbiased and most efficient among all such combined forecasts (i.e., the associated forecast error has a zero mean and the lowest possible variance). If CBIV contains extra information beyond the existing predictor, this optimal weight λ on the CBIV predicted value should be statistically and economically different from zero.

Given the forecast errors associated with CBIV and an existing predictor i, $e^{i}_{t,t+K} = \left(r_{t,t+K} - \hat{r}^{i}_{t,t+K}\right)$ and $e^{CBIV}_{t,t+K} = \left(r_{t,t+K} - \hat{r}^{CBIV}_{t,t+K}\right)$, the forecast error corresponding to $\hat{r}^{*}_{t,t+K}$ is:

$$\epsilon_{t,t+K} = (1-\lambda)e^{i}_{t,t+K} + \lambda e^{CBIV}_{t,t+K}. \tag{32}$$

To estimate the optimal weight λ, we follow Harvey et al. (1998) by running the regression:

$$e^{i}_{t,t+K} = \lambda\left(e^{i}_{t,t+K} - e^{CBIV}_{t,t+K}\right) + \epsilon_{t,t+K}. \tag{33}$$

The hypothesis testing is $H_0\colon \lambda = 0 \ \ vs. \ H_1\colon \lambda > 0$. The *t*-statistic is calculated following Harvey et al. (1998). The results of forecast encompassing tests reported in Table 4 Panel C are consistent with those from the in-sample predictive regressions in Table 3. In all cases (except for CBIV against the Rossi and Timmermann (2015) covariance measure for the one-month forecast horizon), the optimal weight λ is significantly positive. This indicates that CBIV contains substantial new information about future stock market returns beyond existing conditional covariance measures in the literature.

D. Robustness Checks

In this subsection, we conduct a comprehensive examination of the robustness of our main empirical findings to variations in constructing our covariance risk proxy CBIV. First, for each combination of two percentiles $IV_{n,t}$ and $IV_{m,t}$ of the cross-section of individual stock idiosyncratic volatilities (for $(n, m) \in \{1, 2, 3, \ldots, 99\}, n > m$), we construct the corresponding CBIV index by the ratio $\frac{IV_{n,t}}{IV_{m,t}}$ as in equation (17), and use it to forecast future stock market returns. Table 5 presents the empirical results using a matrix where the (n, m) entry reports the absolute value of t-statistics and adjusted R^2's (in parentheses) when regressing future stock market returns on the corresponding CBIV index. To save space, we only list 10 percentiles from 5th to 95th with 10 percentiles per increment (i.e., {5th, 15th, 25th, …, 95th}) and report the forecast performance of CBIV at one-month and six-month horizons. The more general results are similar to Table 5 and available upon request.

[Insert Table 5 approximately here]

Consistent with Proposition 2, the significant stock market return predictability by CBIV is robust to alternative choices of two idiosyncratic volatilities. For example, for 78% of the combinations in Table 5, the absolute value of the *t*-statistic for the corresponding CBIV is greater than 2 when forecasting one-month ahead stock market returns. At the six-month forecast horizon, the coefficient of CBIV comes out significant in 91% of the cases. Similarly, when we pick two out of 99 percentiles to define CBIV, 3,620 out of 4,851 (i.e., 74.62%) cases are significant for the one-month forecast horizon and 4,131 out of 4,851 (i.e., 85.16%) cases are significant for the six-month horizon. Thus, it is a general and robust finding that the ratio of two idiosyncratic volatilities from the whole cross-section of idiosyncratic volatilities possesses a significant predictive power for the stock market returns.

[Insert Table 6 approximately here]

Second, we take equal-weighted idiosyncratic volatility (EWIV) as IV_t^F and either value-weighted idiosyncratic volatility (VWIV) or price-weighted idiosyncratic volatility (PWIV) as IV_t^S. We construct CBIV as the ratio between IV_t^F and IV_t^S. Table 6 shows that EWIV together with VWIV (or PWIV) or the corresponding CBIV have also significant power predicting stock market returns, although the performance is weaker than the combination of IV_t^F and IV_t^S using equations (16) to (20).

IV. ICAPM Specification Tests

In this section, we conduct specification tests of the ICAPM using our covariance risk proxy and provide evidence on the importance of our covariance risk proxy. Our first ICAPM specification test regresses future stock market excess returns on both the conditional market variance and our ICAPM conditional covariance proxy, following equation (14) in Rossi and Timmermann (2015):

$$\sum_{k=1}^{K} \frac{r_{t+k}}{K} \equiv r_{t,t+K} = a + b_1 \times \hat{\sigma}_{M,t}^2 + b_2 \times \hat{\sigma}_{MH,t} + \epsilon_{t,t+K}. \tag{34}$$

We estimate the conditional market variance ($\hat{\sigma}_{M,t}^2$) using the EGARCH model as in Rossi and Timmermann (2015).

[Insert Table 7 approximately here]

Table 7 shows that incorporating the conditional covariance risk in forecasting stock market returns leads to robust evidence on the traditional risk-return tradeoff, namely a positive relationship between the equity risk premium and the market variance. The coefficient of the market variance is statistically significant in Table 7 Panel A. Our estimate of the relative risk aversion (i.e., b_1) falls between 3 and 4, which is reasonable in magnitude and similar to the results in Rossi and Timmermann (2015) using an alternative covariance measure. The coefficients of relative risk aversion for the conditional covariance (i.e., b_2) are also comparable to those in Rossi and Timmermann (2015).[8]

Rossi and Timmermann (2015) show that adding the conditional covariance to the market variance when forecasting one month ahead stock market return increases the regression R^2 from 0.6% to 2.8%. Our Table 7 Panel A provides similar evidence. The improvement is especially significant for the longer forecast horizons. For example, at semiannual (annual) horizon in Table 7 Panel A, the R^2 increases from 5.50% (5.13%) by market variance alone to 13.81% (19.55%) when our ICAPM covariance proxy is included as a predictor. Similar results hold for out-of-sample R^2 tests in Table 7 Panel B. The semiannual (annual) out-of-sample R^2 increases from 2.56% (0.88%) by market variance alone to 11.23% (15.52%) when our covariance proxy CBIV is added as a regressor. These results illustrate the importance of our covariance risk proxy for explaining the equity risk premium as suggested by the ICAPM.

To further evaluate the ICAPM model, we conduct the Ramsey RESET specification test suggested by Rossi and Timmermann (2015). We regress stock market excess returns one month ahead on the conditional variance and covariance and obtain the corresponding residuals. If the ICAPM is correctly specified, these residuals should be uncorrelated with squares of the conditional variance and covariance, or other transformations. We test this implication by

---

[8] For example, when we scale our CBIV to have the same standard deviation of the conditional covariance in Rossi and Timmermann (2015), the estimated coefficient of CBIV in equation (34) is 0.08, which is close to 0.11 in Table 2 of Rossi and Timmermann (2015).

regressing the ICAPM residuals on the squared values of the conditional variance and covariance. Following Rossi and Timmermann (2015), we report a Wald test for their joint significance. The results are provided in Table 7 Panel C. Our results from the Ramsey test are consistent with Rossi and Timmermann (2015). For example, we cannot reject the null hypothesis that ICAPM is correctly specified under our covariance risk proxy CBIV or the covariance measure by Rossi and Timmermann (2015). In contrast, Table 7 Panel C shows that when the covariance measure by Guo and Whitelaw (2006) is used in the Ramsey test, the null hypothesis can be rejected at the 10% level (p-value is 0.0791). This highlights the importance of covariance risk proxy in testing the ICAPM model.

V. Alternative Interpretation of Kelly and Jiang (2014) Tail Index

In this section, we show theoretically and empirically that the conditional covariance risk $\sigma_{MH,t}$ is closely linked to the Kelly and Jiang (2014) tail index. They construct a new measure of tail risk using the cross-section of stock returns under the assumption that individual stock returns are well described by a dynamic power law with a common component. Specifically, the KJ tail index is defined as:

$$\lambda_t^{Hill} = \frac{1}{K_t} \sum_{k=1}^{K_t} ln\left(\frac{\eta_{k,t}}{u_t}\right), \tag{35}$$

where u_t is the 5th percentile of the cross-section distribution of residual stock returns, $\eta_{k,t}$ is the kth daily residual return that falls below an extreme value threshold u_t during month t, and K_t is the total number of these exceedances within month t. The residual returns are obtained by removing the exposures to common return factors under a benchmark factor model such as the CAPM, the Fama-French three or five factor model.[9]

The KJ tail index has a strong predictive power for aggregate stock returns, which Kelly and Jiang (2014) interpret as evidence that firm-level tail risk can influence asset prices under structural models with heavy-tailed distributions of firm-level fundamental growth rate shocks. However, Chapman, Gallmeyer, and Martin (2018) raise some issues about the tail risk interpretation. For example, the KJ tail index has no statistically significant correlation with theoretically motivated measures of macroeconomic uncertainty and systemic risk (e.g., Allen, Bali, and Tang (2012), Bloom (2014), and Jurado, Ludvigson, and Ng (2015)) that should connect to tail risk and thus expected returns. The tail measure also has no apparent conditional correlation with future aggregate dividend growth. Chapman et al. (2018) find that the predictive power of the KJ tail index for stock market returns comes through the discount rate channel and not the cash flow channel. This finding appears inconsistent with a structural model, such as a modified long-run risk model or the disaster risk model, which generates tail outcomes through large real cash flow effects.[10]

We provide an alternative explanation under the ICAPM framework for why KJ's tail index can predict stock market returns without the need to link it to macroeconomic uncertainty and systemic risk or tail risk. Our explanation works through the discount rate channel and does not need any cash flow assumptions. Central to our explanation, we show that the KJ tail index can be capturing the ICAPM covariance risk $\sigma_{MH,t}$. As shown in Proposition 1 equation (9), the

[9] Residual returns in our paper are computed using the CAPM as the benchmark model, although all the empirical results are similar and significant if we use the Fama-French three or five factors model.
[10] These models are motivated through the cash flow channel (see, e.g., Bansal and Yaron (2004), Barro (2006), Bansal and Shaliastovich (2011), Drechsler and Yaron (2011), Gabaix (2012), and Wachter (2013)).

mean of the idiosyncratic returns (i.e., $\eta_{i,t}$ in equation (8)) contains useful information about $\sigma_{MH,t}$, which is a key determinant of the equity risk premium under the ICAPM. The average of $\eta_{i,t}$ across stocks belonging to the lowest five percentiles is used to construct KJ's tail index. Therefore, it is not surprising that there is a close link between the KJ tail index and the ICAPM covariance $\sigma_{MH,t}$, as described by the following Proposition:

Proposition 3. *Under the ICAPM, the Kelly and Jiang (2014) tail index λ_{t+1}^{Hill} is a linear function of the covariance risk $\sigma_{MH,t}$:*

$$\lambda_{t+1}^{Hill} \approx ln\left(\frac{J_t^u}{J_t^{\bar{\eta}}}\right) + \frac{1}{\psi_0}\sigma_{MH,t} + e_{t+1}, \tag{36}$$

$$where \begin{cases} \psi_0 = \frac{E\left(J_t^{\bar{\eta}}\bar{\eta}_{t+1}\right)+E\left(J_t^u u_{t+1}\right)}{2} \\ \bar{\eta}_{t+1} = \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}}\eta_{k,t+1} \end{cases}, \begin{cases} G_t^{\bar{\eta}} = \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}}\gamma_H b_{kM,t} \\ G_t^u = \gamma_H b_{uM,t} \end{cases}, \begin{cases} D_t^{\bar{\eta}} = \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}}\gamma_H b_{kH,t} \\ D_t^u = \gamma_H b_{uH,t} \end{cases},$$

$$and \quad \begin{cases} J_t^{\bar{\eta}} = \frac{D_t^u}{D_t^{\bar{\eta}}G_t^u - D_t^u G_t^{\bar{\eta}}} \\ J_t^u = \frac{D_t^{\bar{\eta}}}{D_t^{\bar{\eta}}G_t^u - D_t^u G_t^{\bar{\eta}}} \end{cases}.$$

Proposition 3 shows that the KJ tail index is approximately proportional to the ICAPM covariance risk $\sigma_{MH,t}$ up to a constant and e_{t+1}, which is a mean zero random variable of true idiosyncratic stock returns (see the proof in the appendix). This provides an alternative theoretic explanation for the ability of the KJ tail index λ_t^{Hill} to predict stock market returns via its relation to the covariance risk $\sigma_{MH,t}$ and the ICAPM pricing relation in equation (4).

Empirically, we first replicate the tail index λ_t^{Hill} and confirm that it is a robust predictor for stock market returns from 1960 to 2022 which extends the finding in Kelly and Jiang (2014). Consistent with Kelly and Jiang (2014), Table 8 Panel A confirms that the return predictive power of λ_t^{Hill} is significant both statistically and economically at horizons from one month to two years.

[Insert Table 8 approximately here]

We then compare the KJ tail index and CBIV, our proxy for the ICAPM covariance risk. Consistent with Proposition 3, the KJ tail index is highly correlated with CBIV, with a correlation of 0.818 over the sample period from 1960 to 2022. Figure 2 illustrates that the two variables closely track each other.

[Insert Figure 2 approximately here]

Table 8 Panel A also shows that both CBIV and the KJ tail index are significant predictors of stock market returns in univariate predictive regressions. But when they are used together in a bivariate predictive regression, the KJ tail index loses its significance to forecast stock market returns while the coefficient of CBIV becomes larger and more significant. Several factors can contribute to the better performance of CBIV compared with the KJ tail index. First, although both the mean and the variance of misspecified idiosyncratic returns contain information about the ICAPM covariance risk (Proposition 1), the first moment (as captured by KJ index) is estimated with more noise than the second moment (CBIV). Second, the KJ tail index only uses information in the tail part of the cross-sectional residual returns, while CBIV utilizes information from the whole cross-section of idiosyncratic volatilities. Thus, CBIV is a better proxy of the ICAPM covariance risk, which explains the superior predictive performance of CBIV for stock market return predictability.

Table 8 Panel B reports the correlations between λ_t^{Hill} and alternative proxies of the ICAPM covariance risk proxy obtained as the ratio of two percentiles $IV_{n,t}$ and $IV_{m,t}$ of the cross-section of stock idiosyncratic volatilities. It is noteworthy that the KJ tail index has very high correlations with CBIVs that are constructed using percentiles that are not in the tail part of the cross-sectional idiosyncratic volatilities. For example, one can choose the two quantiles of 45th and 65th percentiles of the cross-sectional idiosyncratic volatilities to construct CBIV, which is also closely related to the KJ tail index with a correlation of 0.83. In an unreported table using 99 percentiles for different combinations, we find similar evidence that 3,607 out of 4,851 (i.e., 74.36%) combinations have the absolute values of the correlations above 0.6 and most of them are not from the tail part of the cross-sectional idiosyncratic volatilities. Thus, despite the construction of the KJ tail index, its information content about the equity premium does not originate from the tail of the cross-section of stock returns.

Chapman et al. (2018) find that the KJ tail index can forecast the future dividend-price ratio, but it has no link to future dividend growth rates. To reinforce our interpretation of Kelly and Jiang (2014) tail index, we examine the ability of CBIV to forecast cash flow news or discount rate news. Following Huang et al. (2015) and Chapman et al. (2018), we use dividend growth (dividend-price ratio) as a proxy for cash flow news (discount rate news):

$$(37) \qquad \begin{cases} \sum_{k=1}^{K} \frac{DG_{t+k}}{K} \equiv DG_{t,t+K} = a + b \times CBIV_t + \sum_{j=1}^{M} c_j \times X_{j,t} + \epsilon_{t,t+K} \\ \sum_{k=1}^{K} \frac{DP_{t+k}}{K} \equiv DP_{t,t+K} = a + b \times CBIV_t + \sum_{j=1}^{M} c_j \times X_{j,t} + \epsilon_{t,t+K} \end{cases},$$

where DG (DP) stands for dividend growth (divide-price ratio). Following Chapman et al. (2018), when running regressions, we control for the lagged DG, DP, and stock market excess returns. The results are provided in Table 9.

[Insert Table 9 approximately here]

Table 9 shows that CBIV is able to forecast not only the equity risk premium, but also the dividend-price ratio, which is a standard proxy for discount rate news. On the contrary, CBIV does not predict dividend growth rate at any forecast horizons. These results are consistent with the finding of Chapman et al. (2018) since CBIV is highly correlated with the KJ tail index. Our results support a discount rate channel underlying CBIV's return predictive power, which is consistent with our ICAPM covariance risk interpretation of CBIV. Moreover, Kelly and Jiang (2014) tail index could proxy for the ICAPM covariance risk instead of tail risk.

VI. Conclusion

Whether and how idiosyncratic volatility matters for asset pricing is a fundamental and hotly debated topic due to its important implications for both investment theory and practice. We make several novel contributions on this topic, both theoretically and empirically. Grounded in the ICAPM framework, we show that the conditional mean and variance of idiosyncratic stock returns under the one-factor market model can be used to identify the conditional covariance between the market and the hedge portfolio, which predicts the equity risk premium under the ICAPM.

Empirically, we construct a proxy (CBIV) of the ICAPM covariance risk using only the cross-section of idiosyncratic volatilities. We document significant stock market return predictability by CBIV in both in-sample and out-of-sample tests. Our results are robust to variations in the constructions of the covariance risk proxy and to controlling for existing predictors in the literature. Our covariance risk proxy compares well with existing conditional covariance measures. Specifications tests support the ICAPM pricing relations under our covariance risk measure. We also find that two different average idiosyncratic volatilities jointly are strong and robust predictors of aggregate stock returns at both short and long horizons. Our results help reconcile the mixed findings in the literature about the time-series stock market return predictability by average idiosyncratic volatility.

Our paper sheds new light on the economic mechanism underlying the return predictive power of Kelly and Jiang (2014) tail index. We show theoretically their tail index is approximately proportional to the ICAPM covariance risk. Empirically, their tail index and our CBIV track each other closely. While both CBIV and the tail index are significant predictors of stock market returns in univariate predictive regressions, the tail index loses its significance after controlling for CBIV, our proxy of the ICAPM covariance risk. This provides a new interpretation of Kelly and Jiang (2014) tail index.

Our work highlights the idea that idiosyncratic returns under a misspecified factor model contains useful information about the true stochastic discount factor. Further studies along this line should yield more insights.

References

Allen, L.; T. G. Bali; and Y. Tang. "Does Systemic Risk in the Financial Sector Predict Future Economic Downturns?" *Review of Financial Studies*, 25 (2012), 3000-3036.

Amihud, Y. "Illiquidity and Stock Returns: Cross-Section and Time-Series Effects." *Journal of Financial Markets*, 5 (2002), 31-56.

Baker, M., and J. Wurgler. "Investor Sentiment in the Stock Market." *Journal of Economic Perspectives*, 21 (2007), 129-152.

Bali, T. G. "The Intertemporal Relation between Expected Returns and Risk." *Journal of Financial Economics*, 87 (2008), 101-131.

Bali, T. G.; N. Cakici; X. Yan; and Z. Zhang. "Does Idiosyncratic Risk Really Matter?" *Journal of Finance*, 60 (2005), 905-929.

Bali, T. G., and R. F. Engle. "The Intertemporal Capital Asset Pricing Model with Dynamic Conditional Correlations." *Journal of Monetary Economics*, 57 (2010), 377-390.

Bansal, R., and I. Shaliastovich. "Learning and Asset-Price Jumps." *Review of Financial Studies*, 24 (2011), 2738-2780.

Bansal, R., and A. Yaron. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance*, 59 (2004), 1481-1509.

Barro, R. J. "Rare Disasters and Asset Markets in the Twentieth Century." *Quarterly Journal of Economics*, 121 (2006), 823-866.

Bekaert, G.; R. J. Hodrick; and X. Zhang. "Aggregate Idiosyncratic Volatility." *Journal of Financial and Quantitative Analysis*, 47 (2012), 1155-1185.

Bloom, N. "Fluctuations in Uncertainty." *Journal of Economic Perspectives*, 28 (2014), 153-76.

Bollerslev, T.; G. Tauchen; and H. Zhou. "Expected Stock Returns and Variance Risk Premia." *Review of Financial Studies*, 22 (2009), 4463-4492.

Campbell, J. “Intertemporal Asset Pricing without Consumption Data.” *American Economic Review*, 83 (1993), 487-512.

Campbell, J. Y.; M. Lettau; B. G. Malkiel; and Y. Xu. “Have Individual Stocks Become More Volatile? An Empirical Exploration of Idiosyncratic Risk.” *Journal of Finance*, 56 (2001), 1-43.

Campbell, J. Y., and R. J. Shiller. “The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors.” *Review of Financial Studies*, 1 (1988), 195-228.

Campbell, J. Y., and S. B. Thompson. “Predicting Excess Stock Returns Out of Sample: Can Anything Beat the Historical Average?” *Review of Financial Studies*, 21 (2008), 1509-1531.

Chapman, D. A.; M. F. Gallmeyer; and J. S. Martin. “Aggregate Tail Risk and Expected Returns.” *Review of Asset Pricing Studies*, 8 (2018), 36-76.

Chen, Y.; G. W. Eaton; and B. S. Paye. “Micro (Structure) Before Macro? The Predictive Power of Aggregate Illiquidity for Stock Returns and Economic Activity.” *Journal of Financial Economics*, 130 (2018), 48-73.

Chen, Z., and R. Petkova. “Does Idiosyncratic Volatility Proxy for Risk Exposure?” *Review of Financial Studies*, 25 (2012), 2745-2787.

Clark, T. E., and K. D. West. “Approximately Normal Tests for Equal Predictive Accuracy in Nested Models.” *Journal of Econometrics*, 138 (2007), 291-311.

Cox, J. C.; J. E. Ingersoll Jr; and S. A. Ross. “An Intertemporal General Equilibrium Model of Asset Prices.” *Econometrica* 53 (1985), 363-384.

Drechsler, I., and A. Yaron. “What's Vol Got to Do with It.” *Review of Financial Studies*, 24 (2011), 1-45.

Engle, R. “Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models." *Journal of Business & Economic Statistics*, 20 (2002), 339-350.

Fair, R. C., and R. J. Shiller. "Comparing Information in Forecasts from Econometric Models." *American Economic Review* 52 (1990), 375-389.

Gabaix, X. "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance." *Quarterly Journal of Economics*, 127 (2012), 645-700.

Goyal, A., and P. Santa-Clara. "Idiosyncratic Risk Matters!" *Journal of Finance*, 58 (2003), 975-1007.

Goyal, A., and I. Welch. "A Comprehensive Look at the Empirical Performance of Equity Premium Prediction." *Review of Financial Studies*, 21 (2008), 1455-1508.

Goyal, A.; I. Welch; and A. Zafirov. "A Comprehensive 2022 Look at the Empirical Performance of Equity Premium Prediction." *Review of Financial Studies*, 37 (2024), 3490-3557.

Guo, H., and R. Savickas. "Average Idiosyncratic Volatility in G7 Countries." *Review of Financial Studies*, 21 (2008), 1259-1296.

Guo, H., and R. F. Whitelaw. "Uncovering the Risk-Return Relation in the Stock Market." *Journal of Finance*, 61 (2006), 1433-1463.

Han, B., and G. Li. "Information Content of Aggregate Implied Volatility Spread." *Management Science*, 67 (2021), 1249-1269.

Harvey, D.I.; S.J. Leybourne; and P. Newbold. "Tests for Forecast Encompassing." *Journal of Business & Economic Statistics*, 16 (1998), 254-259.

Herskovic, B.; B. Kelly; H. Lustig; and S.V. Nieuwerburgh. "The Common Factor in Idiosyncratic Volatility: Quantitative Asset Pricing Implications." *Journal of Financial Economics*, 119 (2016), 249-283.

Huang, D.; F. Jiang; J. Tu; and G. Zhou. "Investor Sentiment Aligned: A Powerful Predictor of Stock Returns." *Review of Financial Studies*, 28 (2015), 791-837.

Huang, D.; J. Li; and L. Wang. "Are Disagreements Agreeable? Evidence from Information Aggregation." *Journal of Financial Economics*, 141 (2021), 83-101.

Ingersoll, J. E. "Theory of Financial Decision Making." *Rowman & Littlefield*, Vol. 3 (1987).

Jurado, K.; S. C. Ludvigson; and S. Ng. "Measuring Uncertainty." *American Economic Review*, 105 (2015), 1177-1216.

Kandel, S., and R. F. Stambaugh. "On the Predictability of Stock Returns: An Asset-Allocation Perspective." *Journal of Finance*, 51 (1996), 385-424.

Kelly, B., and H. Jiang. "Tail Risk and Asset Prices." *Review of Financial Studies*, 27 (2014), 2841-2871.

Kelly, B., and S. Pruitt. "The Three-Pass Regression Filter: A New Approach to Forecasting Using Many Predictors." *Journal of Econometrics*, 186 (2015), 294-316.

Long Jr, J. B. "Stock Prices, Inflation, and the Term Structure of Interest Rates." *Journal of Financial Economics*, 1 (1974), 131-170.

Martin, I. "What is the Expected Return on the Market?" *Quarterly Journal of Economics*, 132 (2017), 367-433.

Merton, R. C. "An Intertemporal Capital Asset Pricing Model." *Econometrica*, 41 (1973), 867-887.

Newey, W. K., and K. D. West. "A Simple, Positive Semi-Definite, Heteroscedasticity and Autocorrelation Consistent Covariance Matrix." *Econometrica* 55 (1987), 703-708.

Pollet, J. M., and M. Wilson. "Average Correlation and Stock Market Returns." *Journal of Financial Economics*, 96 (2010), 364-380.

Rapach, D. E.; M. C. Ringgenberg; and G. Zhou. "Short Interest and Aggregate Stock Returns." *Journal of Financial Economics*, 121 (2016), 46-65.

Rapach, D. E.; J. K. Strauss; and G. Zhou. “Out-of-Sample Equity Premium Prediction: Combination Forecasts and Links to the Real Economy.” *Review of Financial Studies*, 23 (2010), 821-862.

Rossi, A. G., and A. Timmermann. “Modeling Covariance Risk in Merton's ICAPM.” *Review of Financial Studies*, 28 (2015), 1428-1461.

Wachter, J. A. “Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility?” *Journal of Finance*, 68 (2013), 987-1035.

Wei, S. X., and C. Zhang. “Idiosyncratic Risk Does Not Matter: A Re-Examination of the Relationship between Average Returns and Average Volatilities." *Journal of Banking & Finance*, 29 (2005), 603-621.

FIGURE 1

Time Series of Key Variables

Figure 1 depicts the monthly time series of 10 quantiles of cross-sectional distribution of demeaned idiosyncratic volatility (Ivol), IV_t^F, IV_t^S, and cross-sectional bivariate idiosyncratic volatility (CBIV) from 1960 to 2022. The 10 quantiles of the cross-sectional distribution of stock idiosyncratic volatility are selected from 5th to 95th percentiles with 10 percentiles as the step size (i.e., {5th, 15th, 25th, …, 95th}). Each quantile idiosyncratic volatility is subtracted by the cross-sectional mean of idiosyncratic volatility. IV_t^F, IV_t^S, and CBIV are defined by equations (18) to (20). In particular, for each possible combination of two percentiles $IV_{n,t}$ and $IV_{m,t}$ defined in equation (16) with $(n, m) \in \{1, 2, 3, \dots, 99\}, n > m$, we regress stock market returns next month on the ratio between $IV_{n,t}$ and $IV_{m,t}$ using an expanding window starting from 1950. If the regression coefficient is negative, $IV_{m,t}$ is then adjusted by multiplying it with -1. We take the cross-sectional median of all $IV_{n,t}$'s as of IV_t^F and the cross-sectional median of all adjusted $IV_{m,t}$'s as IV_t^S. CBIV plotted in the bottom panel is defined to be the ratio between of IV_t^F and IV_t^S. The grey areas indicate the National Bureau of Economic Research (NBER) recession periods.

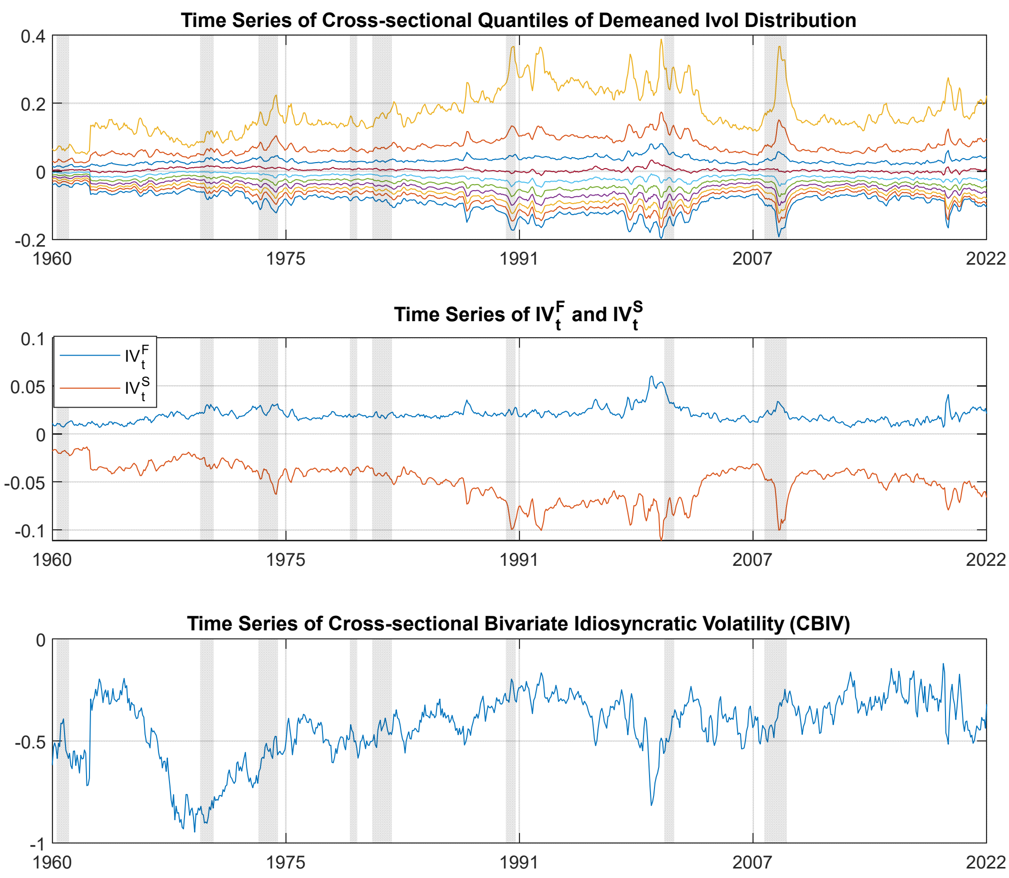

FIGURE 2

Time Series of Tail Risk and Conditional Covariance Risk

Figure 2 compares the monthly time series of the ICAPM covariance risk proxied by the cross-sectional bivariate idiosyncratic volatility (CBIV) and tail risk proposed by Kelly and Jiang (2014). The tail risk measure (KJ Tail Index) is constructed following Kelly and Jiang (2014). Cross-sectional bivariate idiosyncratic volatility (CBIV) is the ratio of two idiosyncratic volatilities from the cross-section of stock idiosyncratic volatilities as given in equation (20). For ease of comparison, we scale the value of both variables to have zero mean and one standard deviation. The sample period is from 1960 to 2022. The grey areas indicate the National Bureau of Economic Research (NBER) recession periods.

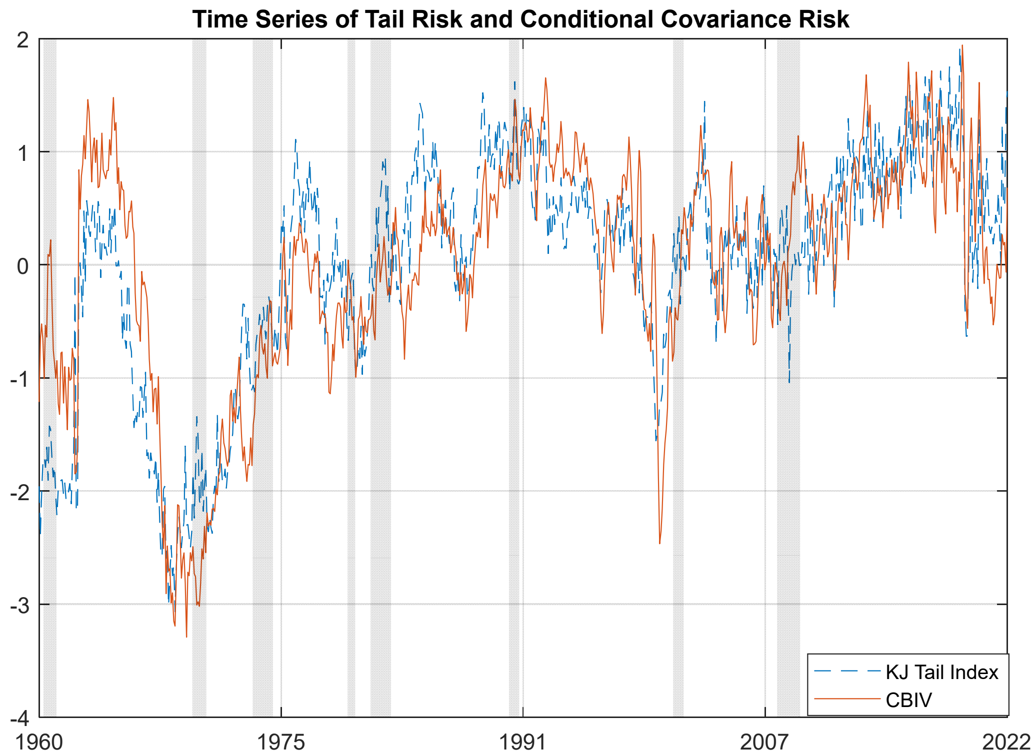

TABLE 1

Summary Statistics of Key Variables

Table 1 reports the descriptive statistics for the monthly time series of some key variables in our study including our covariance risk measures obtained from the cross-section of stock idiosyncratic volatilities (IV_t^F, IV_t^S, and CBIV as given in equations (18) to (20)), stock market volatility (SMV), and stock market excess returns (MKTRT) in Panel A, as well as the correlations among them and other important predictors in Panel B. The summary statistics include mean, standard deviation (STD), and autocorrelations with various numbers of lagged months. MKTRF is obtained from Kenneth French's website. SMV is calculated using the past 30-day daily MKTRF. GW Index and RT Index are the alternative conditional covariance measures proposed by Guo and Whitelaw (2006) and Rossi and Timmermann (2015). KJ Index is the tail measure proposed by Kelly and Jiang (2014), while HLW Disp is an aggregate disagreement measure constructed by Huang et al. (2021). All variables are expressed as monthly values and computed at monthly frequency.

Panel A. Summary Statistics

			Autocorrelation at Lag (Number of Months)				
Variable	**Mean**	**STD**	**1**	**3**	**6**	**12**	**24**
IV_t^F	0.020	0.008	0.954	0.826	0.757	0.610	0.382
IV_t^S	-0.049	0.019	0.982	0.910	0.827	0.772	0.649
CBIV	-0.427	0.158	0.958	0.869	0.824	0.737	0.559
SMV	0.040	0.025	0.752	0.404	0.311	0.169	0.074
MKTRF	0.005	0.045	0.050	0.028	-0.059	0.022	-0.0003

Panel B. Pearson Correlation Matrix

Variable	$\boldsymbol{IV_t^F}$	$\boldsymbol{IV_t^S}$	**CBIV**	**SMV**	**GW Index**	**RT Index**	**KJ Index**
IV_t^S	-0.582						
CBIV	-0.356	-0.475					
SMV	0.442	-0.340	-0.042				
GW Index	0.177	-0.406	0.282	-0.033			
RT Index	-0.058	-0.117	0.215	0.025	0.167		
KJ Index	-0.159	-0.472	0.818	0.013	0.223	0.195	
HLW Disp	0.291	-0.200	-0.056	0.260	0.013	-0.021	-0.061

TABLE 2

In-sample Predictive Regressions

This table reports the results of univariate and bivariate monthly predictive time-series regressions in equation (21). The dependent variables are average monthly stock market excess returns (MKTRF) over the relevant forecast horizons K (in number of months). All predictors are normalized to have zero mean and one standard deviation. "*b*" is the slope coefficient on the predictor multiplied by 100. When $K > 1$, to adjust for the overlapping dependent variable, the *t*-stat is computed using the GMM standard errors with $K - 1$ Newey-West lag correction. The sample period is from 1960 to 2022 in Panel A and specified in each row in Panel B.

Panel A. Univariate and Bivariate Predictive Regression by IV_t^F and IV_t^S

		K=1			K=2			K=3		
Predictor	**Coefficient**	**I**	**II**	**III**	**I**	**II**	**III**	**I**	**II**	**III**
IV_t^F	b	-0.114		-0.378	-0.110		-0.382	-0.156		-0.455
	t-stat	(-0.62)		(-1.76)	(-0.70)		(-2.08)	(-1.10)		(-2.70)
IV_t^S	b		-0.235	-0.455		-0.246	-0.469		-0.250	-0.515
	t-stat		(-1.38)	(-2.34)		(-1.62)	(-2.74)		(-1.79)	(-3.29)
	adj. R^2 (%)	-0.07	0.14	0.48	-0.02	0.44	1.23	0.22	0.77	2.63

Predictor	**Coefficient**		**K=6**			**K=12**			**K=24**	
IV_t^F	b	-0.233		-0.556	-0.230		-0.534	-0.200		-0.463
	t-stat	(-1.80)		(-3.69)	(-1.72)		(-3.65)	(-1.48)		(-3.46)
IV_t^S	b		-0.233	-0.556		-0.215	-0.523		-0.180	-0.450
	t-stat		(-1.72)	(-3.69)		(-1.50)	(-3.62)		(-1.23)	(-3.71)
	adj. R^2 (%)	1.38	1.37	6.93	2.81	2.45	12.88	4.89	3.95	21.59

Panel B. Univariate Predictive Regression by Cross-sectional Bivariate Idiosyncratic Volatility (CBIV)

Predictor	**Coefficient**	**K=1**	**K=2**	**K=3**	**K=6**	**K=12**	**K=24**
CBIV	b	0.455	0.443	0.445	0.419	0.396	0.271
1960-1991	t-stat	(2.04)	(2.23)	(2.36)	(2.24)	(2.15)	(1.99)
	adj. R^2 (%)	0.74	1.50	2.40	4.24	8.39	10.67
CBIV	b	0.305	0.325	0.444	0.604	0.582	0.528
1992-2022	t-stat	(1.25)	(1.56)	(2.35)	(3.94)	(3.86)	(4.16)
	adj. R^2 (%)	0.21	0.79	2.80	10.47	17.91	28.51
CBIV	b	0.399	0.411	0.449	0.482	0.461	0.389
1960-2022	t-stat	(2.51)	(2.90)	(3.40)	(3.65)	(3.53)	(3.67)
	adj. R^2 (%)	0.66	1.48	2.78	6.33	11.80	19.00

TABLE 3

Multiple Regressions Controlling for Existing Predictors

Table 3 reports the results of multiple predictive regressions in equation (22) where the regressors include our covariance measures based on the cross-section of stock idiosyncratic volatilities (IV_t^F, IV_t^S, and CBIV) and various stock market return predictors in the literature as controls. The dependent variables are average monthly stock market excess returns (MKTRF), and the forecast horizons are six months ($K = 6$) and one year ($K = 12$). GW PCA 1/2/3 are the first three principal components of the 14 predictors based on Goyal and Welch (2008). Other control variables include stock market volatility (SMV), alternative conditional covariance measures GW Index by Guo and Whitelaw (2006) and RT Index by Rossi and Timmermann (2015), aggregate disagreement measure (HLW Disp, Huang et al. (2021)), average stock correlation (Pollet and Wilson (2010)), aggregate stock illiquidity (Chen et al. (2018)), and aggregate investor sentiment index (PLS Sentiment, Huang et al. (2015)). The sample period is from 1970 to 2019 (due to the availability of some control variables). We normalize all predictors to have zero mean and one standard deviation. *K* represents the forecast horizon in the number of months. "*b*" is the slope coefficient on the corresponding predictor multiplied by 100. When $K > 1$, to adjust for the overlapping dependent variable, the *t*-stat is computed using the GMM standard errors with $K - 1$ Newey-West correction.

Predictor	Coefficient	K=6	K=12	K=6	K=12
CBIV	b			0.441	0.487
	t-stat			(2.79)	(3.12)
IV_t^F	b	-0.303	-0.387		
	t-stat	(-1.94)	(-2.67)		
IV_t^S	b	-0.546	-0.522		
	t-stat	(-3.33)	(-3.23)		
SMV	b	-0.216	-0.113	-0.056	-0.011
	t-stat	(-1.07)	(-1.14)	(-0.35)	(-0.11)
GW Index	b	-0.061	-0.015	-0.009	0.006
	t-stat	(-0.52)	(-0.16)	(-0.08)	(0.06)
RT Index	b	0.284	0.263	0.273	0.254
	t-stat	(1.86)	(2.05)	(1.83)	(2.07)
GW PCA 1	b	-0.060	-0.133	0.032	-0.079
	t-stat	(-0.29)	(-0.73)	(0.17)	(-0.43)
GW PCA 2	b	-0.279	-0.181	-0.283	-0.178
	t-stat	(-1.36)	(-1.41)	(-1.36)	(-1.38)
GW PCA 3	b	-0.020	0.004	-0.039	-0.011
	t-stat	(-0.19)	(0.08)	(-0.36)	(-0.18)
HLW Disp	b	-0.403	-0.256	-0.397	-0.261
	t-stat	(-2.71)	(-2.10)	(-2.72)	(-2.15)
Average	b	0.419	0.276	0.258	0.168
Correlation	t-stat	(2.36)	(2.85)	(1.85)	(1.71)
Aggregate	b	0.313	0.178	0.421	0.279
Illiquidity	t-stat	(1.28)	(0.98)	(1.66)	(1.47)
PLS	b	-0.227	-0.163	-0.186	-0.130
Sentiment	t-stat	(-1.50)	(-1.09)	(-1.20)	(-0.85)
	adj. R^2 (%)	18.91	27.26	18.02	26.76

TABLE 4

Out-of-sample Tests

We split the test sample into two parts: 1960 to 1979 as the in-sample estimation period and 1980 to 2022 as the out-of-sample performance evaluation period to compute the statistics in the table. The forecast targets are average monthly stock market excess returns (MKTRF). In Panel A, the out-of-sample R_{OS}^2 statistics are calculated based on equation (25). The predictors are specified in the first column. "$IV_t^F + IV_t^S$" stands for using both IV_t^F and IV_t^S together in a bivariate regression to forecast MKTRF. The *z*-stat is computed based on Clark and West (2007). In Panel B, the Sharpe ratio and certainty equivalent return (CER) are specified in equations (28) and (29) respectively. "Historical Average" stands for using the historical average of MKTRF recursively since 1960 to forecast MKTRF. *K* represents the forecast horizon in the number of months. All the statistics in Panel B are expressed as annual values. In Panel C, the forecast encompassing test is $H_0: \lambda = 0; H_1: \lambda > 0$, where λ measures the extra information from CBIV relative to an alternative predictor i. It is defined as the weight on the predicted market return based on CBIV in the optimal combined forecast: $\hat{r}_{t,t+K}^{*} = (1-\lambda)\hat{r}_{t,t+K}^{i} + \lambda\hat{r}_{t,t+K}^{CBIV}$. The optimal weight λ is estimated from regression $e_{t,t+K}^{i} = \lambda(e_{t,t+K}^{i} - e_{t,t+K}^{CBIV}) + \epsilon_{t,t+K}$, following Harvey et al. (1998). The *t*-stat of λ is calculated from the *t*-distribution of one-tail hypothesis test. *, **, and *** indicate significance at the 10%, 5%, and 1% level, respectively.

Panel A. Out-of-sample R^2 Statistics

Predictor	Statistic	K=1	K=2	K=3	K=6	K=12	K=24
IV_t^F	R_{OS}^2 (%)	-0.309	-0.583	-0.598	-0.736	-3.020	-12.547
	z-stat	-0.072	0.039	0.192	0.152	-0.429	-1.075
IV_t^S	R_{OS}^2 (%)	-0.458	-0.823	-0.832	-0.318	-0.487	-4.743
	z-stat	1.238	1.401	1.572	1.553	1.275	0.820
$IV_t^F + IV_t^S$	R_{OS}^2 (%)	-0.370	-0.420	0.783	5.480	8.847	7.437
	z-stat	1.532	1.835	2.460	3.748	4.290	3.350
CBIV	R_{OS}^2 (%)	0.524	1.254	2.801	7.456	13.676	23.141
	z-stat	1.775	2.240	3.085	4.185	4.647	5.497

Panel B. Optimal Portfolio Sharpe Ratio and Certainty Equivalent Return (CER) Gain

Predictor	Statistic	K=1	K=2	K=3	K=6	K=12	K=24
IV_t^F	Sharpe Ratio	0.411	0.383	0.466	0.475	0.479	0.491
	CER Gain (%)	-0.017	-0.191	0.338	0.493	0.215	0.113
IV_t^S	Sharpe Ratio	0.433	0.426	0.462	0.489	0.525	0.583
	CER Gain (%)	-0.183	-0.226	0.126	0.599	0.835	1.437
$IV_t^F + IV_t^S$	Sharpe Ratio	0.471	0.466	0.571	0.573	0.597	0.612
	CER Gain (%)	0.600	0.711	2.190	2.225	2.416	2.202
CBIV	Sharpe Ratio	0.512	0.490	0.582	0.549	0.561	0.591
	CER Gain (%)	1.539	1.450	2.317	1.769	1.687	1.918
Historical Average	Sharpe Ratio	0.410	0.392	0.443	0.443	0.470	0.491
	CER Gain (%)	-	-	-	-	-	-

Panel C. Comparison with Existing Conditional Covariance Measures

Out-of-Sample R^2 Statistic (%)				Forecast Encompassing Test (λ)			
Predictor	K=1	K=6	K=12	Predictor	K=1	K=6	K=12
CBIV	0.524**	7.456**	13.676***	CBIV	-	-	-
GW Index	-0.387	-1.419	-3.007	GW Index	1.236***	1.525***	1.503***
RT Index	0.619**	0.593*	0.836*	RT Index	0.402	1.369***	1.365***

TABLE 5

Predictive Regressions using Alternative Constructions of CBIV

Table 5 reports the results of the univariate predictive regressions of average monthly stock market excess returns (MKTRF) on various covariance risk proxies CBIV obtained as the ratio between two quantiles in the cross-section of stock idiosyncratic volatilities. Each matrix element is a pair of the absolute value of the t-statistic and the corresponding adjusted R^2 (in parentheses, expressed as percentage) for the univariate regressions with the regressor $CBIV_t = \frac{IV_{n,t}}{IV_{m,t}}$ where n (resp. m) is labeled by the row (resp. column). In Panel A (B), the forecast horizon K is one-month (six-month). All the t-statistics are computed using the GMM standard errors with $K - 1$ Newey-West correction. The test sample is monthly data from 1960 to 2022.

Panel A. One-month Forecast Horizon

Percentile	5th	15th	25th	35th	45th	55th	65th	75th	85th
15th	1.64								
	(0.28)								
25th	2.02	2.32							
	(0.44)	(0.60)							
35th	2.29	2.52	2.53						
	(0.57)	(0.69)	(0.66)						
45th	2.40	2.57	2.59	2.51					
	(0.65)	(0.73)	(0.73)	(0.71)					
55th	2.51	2.63	2.64	2.64	2.58				
	(0.70)	(0.76)	(0.77)	(0.78)	(0.74)				
65th	2.51	2.58	2.60	2.61	2.51	2.00			
	(0.66)	(0.71)	(0.73)	(0.75)	(0.70)	(0.16)			
75th	2.17	2.45	2.54	2.61	2.51	2.12	0.67		
	(0.45)	(0.61)	(0.69)	(0.73)	(0.71)	(0.17)	(-0.06)		
85th	0.46	0.99	1.94	2.46	2.47	2.15	0.71	2.56	
	(-0.11)	(-0.00)	(0.39)	(0.65)	(0.70)	(0.17)	(-0.05)	(0.66)	
95th	2.41	2.17	1.46	0.30	2.22	2.18	0.80	2.56	2.26
	(0.61)	(0.49)	(0.16)	(-0.12)	(0.56)	(0.16)	(-0.04)	(0.64)	(0.49)

Panel B. Six-month Forecast Horizon

Percentile	5th	15th	25th	35th	45th	55th	65th	75th	85th
15th	1.91								
	(2.05)								
25th	2.55	3.25							
	(3.60)	(5.45)							
35th	3.03	3.48	3.45						
	(4.67)	(5.92)	(5.41)						
45th	3.42	3.75	3.80	3.82					
	(5.58)	(6.43)	(6.33)	(6.31)					
55th	3.74	3.88	3.87	3.78	3.53				
	(6.15)	(6.59)	(6.55)	(6.47)	(6.00)				
65th	3.97	3.98	3.96	3.85	3.64	2.11			
	(6.64)	(6.76)	(6.81)	(6.71)	(6.30)	(1.40)			
75th	3.19	3.36	3.52	3.49	3.40	2.08	3.66		
	(4.65)	(5.32)	(5.87)	(5.98)	(5.83)	(1.26)	(0.35)		
85th	0.44	1.12	2.69	3.42	3.58	2.10	3.60	4.13	
	(-0.02)	(0.59)	(3.28)	(5.01)	(5.73)	(1.30)	(0.34)	(6.43)	
95th	3.53	3.39	2.50	0.11	3.00	2.10	3.43	4.03	3.40
	(5.60)	(5.41)	(2.82)	(-0.13)	(4.22)	(1.26)	(0.31)	(6.20)	(4.79)

TABLE 6

Predictive Regressions using Weighted Average Idiosyncratic Volatilities

This table reports the results of univariate and bivariate predictive time-series regressions as in equation (21). The dependent variables are average monthly stock market excess returns (MKTRF) over the relevant forecast horizons *K* (in number of months). The predictors are equal-weighted idiosyncratic volatility (EWIV), value-weighted idiosyncratic volatility (VWIV) using stock market capitalization as weight, price-weighted idiosyncratic volatility (PWIV) using stock price as weight, and the corresponding covariance risk (CBIV) proxy obtained by taking the ratio between EWIV and VWIV (or PWIV). All predictors are normalized to have zero mean and one standard deviation. "*b*" is the slope coefficient on the predictor multiplied by 100. When $K > 1$, to adjust for the overlapping dependent variable, the *t*-stat is computed using the GMM standard errors with $K - 1$ Newey-West lag correction. The test sample is monthly data from 1960 to 2022.

Panel A. Bivariate Predictive Regressions

		K=1			K=6			K=12		
Predictor	**Coefficient**	**I**	**II**	**III**	**I**	**II**	**III**	**I**	**II**	**III**
EWIV	b	0.109		0.814	0.091		0.685	0.052		0.597
	t-stat	(0.62)		(2.91)	(0.69)		(3.01)	(0.37)		(2.82)
VWIV	b		-0.149	-0.836		-0.126	-0.704		-0.140	-0.644
	t-stat		(-0.75)	(-2.53)		(-0.85)	(-2.76)		(-0.91)	(-2.75)
	adj. R^2 (%)	-0.07	-0.02	0.80	0.10	0.31	3.93	0.02	0.97	6.55
Predictor	**Coefficient**		**K=1**			**K=6**			**K=12**	
EWIV	b	0.109		1.025	0.091		0.992	0.052		1.042
	t-stat	(0.62)		(2.55)	(0.69)		(3.35)	(0.37)		(3.99)
PWIV	b		-0.079	-1.009		-0.093	-0.995		-0.146	-1.094
	t-stat		(-0.44)	(-2.43)		(-0.72)	(-3.38)		(-1.05)	(-4.51)
	adj. R^2 (%)	-0.07	-0.10	0.69	0.10	0.10	4.79	0.02	1.06	11.69

Panel B. Univariate Predictive Regressions

Predictor	**Coefficient**	**K=1**	**K=2**	**K=3**	**K=6**	**K=12**	**K=24**
CBIV=	b	0.398	0.388	0.380	0.345	0.289	0.234
$EWIV/VWIV$	t-stat	(2.71)	(2.97)	(2.95)	(2.77)	(2.46)	(2.29)
	adj. R^2 (%)	0.66	1.30	1.95	3.17	4.49	6.64
CBIV=	b	0.444	0.450	0.448	0.431	0.433	0.393
$EWIV/PWIV$	t-stat	(2.72)	(3.12)	(3.37)	(3.38)	(3.53)	(3.93)
	adj. R^2 (%)	0.85	1.79	2.76	5.00	10.34	19.36

TABLE 7

ICAPM Specification Test

Panel A reports the results of ICAPM specification test in equation (34) that regresses the stock market excess returns on both the conditional market variance and our covariance risk proxy CBIV. The conditional market variance is estimated using the EGARCH model based on Rossi and Timmermann (2015). The independent variables are all annualized and based on raw values, while the dependent variables are average annualized monthly stock market excess returns over various future horizons K (in number of months). "b" is the slope coefficient on the corresponding predictor. When $K > 1$, to adjust for the overlapping dependent variable, the t-stat is computed using the GMM standard errors with $K - 1$ Newey-West correction. In Panel B, we split the test sample (monthly data from 1960 to 2022) into two parts: 1960 to 1979 as the in-sample estimation period and 1980 to 2022 as the out-of-sample performance evaluation period to compute the statistics in the table. The forecast targets are stock market excess returns over various future horizons K. The out-of-sample R^2_{OS} statistics are calculated based on equation (25). The predictors are specified in the first column. "EGARCH+CBIV" stands for using both EGARCH and CBIV together in a bivariate regression to forecast stock market excess returns. The z-stat is computed based on Clark and West (2007). Panel C reports the results of the Ramsey RESET specification test following Rossi and Timmermann (2015). We first regress stock market excess returns one month ahead on the market variance and the covariance risk proxy. We then project the residuals of the ICAPM regression on the squared values of the regressors (i.e., conditional market variance and covariance risk) and conduct a Wald test for the null that they are jointly insignificant. Each row of Panel C corresponds to using the market variance and one proxy of covariance risk (our CBIV, GW Index by Guo and Whitelaw (2006), or RT Index from Rossi and Timmermann (2015)) as the regressors in Ramsey RESET specification test of ICAPM.

Panel A. In-sample Predictive Regression

		K=1		K=6		K=12	
Predictor	**Coefficient**	**I**	**II**	**I**	**II**	**I**	**II**
EGARCH	b	3.383	3.860	3.340	3.915	2.270	2.802
	t-stat	(2.30)	(2.60)	(4.11)	(5.11)	(3.15)	(4.29)
CBIV	b		0.359		0.423		0.391
	t-stat		(2.92)		(4.39)		(4.19)
	adj. R^2 (%)	0.92	1.88	5.50	13.81	5.13	19.55

Panel B. Out-of-sample R^2

Predictor	**Statistic**	**K=1**	**K=6**	**K=12**
EGARCH	R^2_{OS} (%)	0.587	2.563	0.875
	z-stat	1.547	2.025	1.229
CBIV	R^2_{OS} (%)	0.524	7.456	13.676
	z-stat	1.775	4.185	4.647
EGARCH + CBIV	R^2_{OS} (%)	1.192	11.225	15.518
	z-stat	2.167	3.718	3.789

Panel C. Ramsey RESET Test

Predictor	**Wald-Test**	**p-value**
CBIV	0.3481	0.5552
GW Index	3.0833	0.0791
RT Index	0.1732	0.6773

TABLE 8

Tail Risk and Conditional Covariance Risk

Panel A reports the results of univariate and bivariate monthly time-series predictive regressions where the dependent variables are average monthly stock market excess returns (MKTRF) over the relevant forecast horizons. λ_t^{Hill} is the Kelly and Jiang (2014) tail index (KJ index) given in equation (35). CBIV is our covariance risk proxy given in equations (16) to (20). All predictors are normalized to have zero mean and one standard deviation. K represents the forecast horizon in the number of months. "b" is the coefficient on the predictor multiplied by 100. When $K > 1$, to adjust for the overlapping dependent variable, the t-stat is computed using the GMM standard errors with $K - 1$ Newey-West lag correction. Panel B presents the absolute value of the Pearson correlation coefficient between the KJ tail index and the covariance risk proxy CBIV obtained as the ratio of two percentiles of the cross-section of stock idiosyncratic volatilities $IV_{n,t}$ and $IV_{m,t}$ where n (resp. m) is labeled by the row (resp. column). The sample period is from 1960 to 2022.

Panel A. Comparison between KJ Tail Index and the Covariance Risk CBIV

Predictor	**Coefficient**	**K=1**	**K=2**	**K=3**	**K=6**	**K=12**	**K=24**
CBIV	b	0.399	0.411	0.449	0.482	0.461	0.389
Only	t-stat	(2.51)	(2.90)	(3.40)	(3.65)	(3.53)	(3.67)
	adj. R^2 (%)	0.66	1.48	2.78	6.33	11.80	19.00
λ_t^{Hill}	b	0.271	0.309	0.336	0.341	0.360	0.370
Only	t-stat	(1.82)	(2.32)	(2.60)	(2.73)	(3.14)	(4.09)
	adj. R^2 (%)	0.23	0.77	1.49	3.08	7.11	17.01
CBIV	b	0.536	0.482	0.529	0.620	0.509	0.267
	t-stat	(1.90)	(2.08)	(2.46)	(2.82)	(2.43)	(1.76)
λ_t^{Hill}	b	-0.168	-0.086	-0.099	-0.169	-0.059	0.149
	t-stat	(-0.63)	(-0.40)	(-0.47)	(-0.82)	(-0.34)	(1.30)
	adj. R^2 (%)	0.58	1.37	2.70	6.47	11.74	19.79

Panel B. Pearson Correlation Coefficients between KJ Tail Index and Alternative Covariance Proxies

Percentile	**5th**	**15th**	**25th**	**35th**	**45th**	**55th**	**65th**	**75th**	**85th**
15th	0.55								
25th	0.64	0.69							
35th	0.72	0.75	0.75						
45th	0.76	0.78	0.79	0.79					
55th	0.80	0.81	0.82	0.82	0.83				
65th	0.80	0.81	0.82	0.82	0.83	0.52			
75th	0.68	0.76	0.80	0.82	0.83	0.51	0.02		
85th	0.13	0.35	0.63	0.75	0.81	0.51	0.02	0.75	
95th	0.76	0.70	0.52	0.02	0.65	0.50	0.02	0.74	0.70

TABLE 9

Economic Channels Underlying the Return Predictive Power of Covariance Risk

This table reports the results of multiple predictive regressions in equation (37). The dependent variable is dividend growth (DP) over various future horizons in Panel A and dividend-price ratio (DG) over various future horizons in Panel B. The independent variables include our lagged covariance risk proxy CBIV and lagged stock market excess returns (MKTRF). The sample period is from 1960 to 2022. We normalize all predictors to have zero mean and one standard deviation. *K* represents the forecast horizon in the number of months. "b" is the slope coefficient on the corresponding predictor and expressed in percent (raw value multiplied by 100). When $K > 1$, to adjust for the overlapping dependent variable, the *t*-stat is computed using the GMM standard errors with $K - 1$ Newey-West correction.

Panel A. Forecast Discount Rate News (Dividend-Price Ratio)

Predictor	Coefficient	K=1	K=2	K=3	K=6	K=12	K=24
CBIV	b	-0.412	-0.647	-0.904	-1.683	-3.043	-4.900
	t-stat	(-2.66)	(-3.19)	(-3.59)	(-4.06)	(-3.97)	(-3.62)
DG	b	0.773	1.132	1.478	2.418	3.793	4.633
	t-stat	(4.67)	(5.61)	(5.94)	(5.71)	(5.50)	(4.24)
DP	b	39.803	39.636	39.449	38.862	37.770	36.213
	t-stat	(230.42)	(183.87)	(145.20)	(83.86)	(43.85)	(23.83)
MKTRF	b	-0.112	0.010	0.032	-0.009	0.249	0.867
	t-stat	(-0.55)	(0.04)	(0.13)	(-0.03)	(0.84)	(2.44)
	adj. R^2 (%)	98.84	98.54	98.22	97.14	95.02	91.21

Panel B. Forecast Cash Flow News (Dividend Growth)

Predictor	Coefficient	K=1	K=2	K=3	K=6	K=12	K=24
CBIV	b	0.006	0.011	0.015	0.023	0.046	0.076
	t-stat	(0.83)	(1.04)	(1.14)	(1.22)	(1.77)	(2.50)
DG	b	0.505	0.482	0.458	0.416	0.329	0.177
	t-stat	(36.68)	(26.54)	(18.83)	(10.64)	(7.36)	(5.45)
DP	b	-0.009	-0.012	-0.014	-0.016	-0.009	0.031
	t-stat	(-1.12)	(-1.11)	(-1.06)	(-0.89)	(-0.31)	(0.62)
MKTRF	b	0.008	0.015	0.024	0.037	0.053	0.050
	t-stat	(1.03)	(1.60)	(2.05)	(2.23)	(2.13)	(2.15)
	adj. R^2 (%)	84.54	80.63	76.04	68.72	51.19	25.70

Idiosyncratic Volatility and the ICAPM Covariance Risk

Appendix

A. Proof of Proposition 1

Assume the true stock excess return generating process for security *i*:

$$R_{i,t+1} = \beta_{iM,t} R_{M,t+1} + \beta_{iH,t} R_{H,t+1} + \varepsilon_{i,t+1}, \tag{A.1}$$

where $R_{M,t+1}$ is the market excess return, $R_{H,t+1}$ is the return of the hedge portfolio, $\varepsilon_{i,t+1}$ is the true idiosyncratic shock for security *i* (which has mean zero by definition) and $\beta_{iM,t}$ and $\beta_{iH,t}$ are the corresponding loadings:

$$\beta_{iM,t} = \frac{\sigma_{iM,t}\sigma_{H,t}^2 - \sigma_{iH,t}\sigma_{MH,t}}{\sigma_{H,t}^2\sigma_{M,t}^2 - \sigma_{MH,t}^2}, \quad \beta_{iH,t} = \frac{\sigma_{iH,t}\sigma_{M,t}^2 - \sigma_{iM,t}\sigma_{MH,t}}{\sigma_{H,t}^2\sigma_{M,t}^2 - \sigma_{MH,t}^2}. \tag{A.2}$$

Suppose that econometricians only use a simple (misspecified) CAPM model to estimate the idiosyncratic risk:

$$R_{i,t+1} = b_{iM,t} R_{M,t+1} + \eta_{i,t+1}, \tag{A.3}$$

where $b_{iM,t}$ is the market beta for security i defined as $b_{iM,t} = \frac{\sigma_{iM,t}}{\sigma_{M,t}^2}$ and $\eta_{i,t+1}$ is the misspecified idiosyncratic return. It can be shown that:

$$\beta_{iM,t} - b_{iM,t} = \frac{\sigma_{iM,t}\sigma_{H,t}^2 - \sigma_{iH,t}\sigma_{MH,t}}{\sigma_{H,t}^2\sigma_{M,t}^2 - \sigma_{MH,t}^2} - \frac{\sigma_{iM,t}}{\sigma_{M,t}^2} = -\beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma_{M,t}^2}. \tag{A.4}$$

The conditional mean of the misspecified stock idiosyncratic return is given by:

$$(A.5) \qquad E_t\left(\eta_{i,t+1}\right) = \mu_{i,t} - b_{iM,t}\mu_{M,t}$$
$$= \beta_{iM,t}\mu_{M,t} + \beta_{iH,t}\mu_{H,t} - \left(\beta_{iM,t} + \beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma^2_{M,t}}\right)\mu_{M,t}$$
$$= \beta_{iH,t}\mu_{H,t} - \beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma^2_{M,t}}\mu_{M,t}.$$

Based on the ICAPM, the risk premia associated with the market portfolio and the hedge portfolio are given by:

$$(A.6) \qquad \begin{cases} \mu_{M,t} = \gamma_M\sigma^2_{M,t} + \gamma_H\sigma_{MH,t} \\ \mu_{H,t} = \gamma_M\sigma_{MH,t} + \gamma_H\sigma^2_{H,t} \end{cases}.$$

Equations (A.5) and (A.6) imply the following relation:

$$(A.7) \qquad E_t(\eta_{i,t+1}) = \beta_{iH,t}(\gamma_M\sigma_{MH,t} + \gamma_H\sigma^2_{H,t}) - \beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma^2_{M,t}}(\gamma_M\sigma^2_{M,t} + \gamma_H\sigma_{MH,t})$$
$$= \gamma_H\beta_{iH,t}\frac{\sigma^2_{H,t}\sigma^2_{M,t} - \sigma^2_{MH,t}}{\sigma^2_{M,t}}$$
$$= \gamma_H\frac{\sigma_{iH,t}}{\sigma^2_{H,t}}\sigma^2_{H,t} - \gamma_H\frac{\sigma_{iM,t}}{\sigma^2_{M,t}}\sigma_{MH,t}$$
$$= \gamma_H b_{iH,t}\sigma^2_{H,t} - \gamma_H b_{iM,t}\sigma_{MH,t}\,.$$

Similarly, the conditional variance of the misspecified firm idiosyncratic return is given by:

$$(A.8) \qquad Var_t(\eta_{i,t+1}) = Var_t(R_{i,t+1} - b_{iM,t}R_{M,t+1})$$
$$= \beta^2_{iH,t}\left(\frac{\sigma_{MH,t}}{\sigma^2_{M,t}}\right)^2\sigma^2_{M,t} + \beta^2_{iH,t}\sigma^2_{H,t} - 2\beta_{iH,t}\beta_{iH,t}\frac{\sigma^2_{MH,t}}{\sigma^2_{M,t}} + \sigma^2_{\varepsilon_i,t}$$
$$= \beta_{iH,t}\frac{\sigma_{iH,t}}{\sigma^2_{H,t}}\sigma^2_{H,t} - \beta_{iH,t}\frac{\sigma_{iM,t}}{\sigma^2_{M,t}}\sigma_{MH,t} + \sigma^2_{\varepsilon_i,t}$$
$$= \beta_{iH,t}b_{iH,t}\sigma^2_{H,t} - \beta_{iH,t}b_{iM,t}\sigma_{MH,t} + \sigma^2_{\varepsilon_i,t}\,.$$

This completes the proof.

B. Proof of Corollary 1.1

Based on Proposition 1, the average idiosyncratic variances IV_t^F and IV_t^S corresponding to two different sets of weights, $w_{i,t}^F$ and $w_{i,t}^S$, can be expressed as different combinations of $\sigma_{H,t}^2$ and $\sigma_{MH,t}$:

$$(A.9) \qquad IV_t^F = \sum_{i=1}^{N_t} w_{i,t}^F Var_t(\eta_{i,t+1}) = A_t^F \sigma_{H,t}^2 - B_t^F \sigma_{MH,t} + \Omega_t^{\mathrm{F}},$$

where $A_t^F = \sum_{i=1}^{N_t} w_{i,t}^F \beta_{iH,t} b_{iH,t}$, $B_t^F = \sum_{i=1}^{N_t} w_{i,t}^F \beta_{iH,t} b_{iM,t}$, $\Omega_t^{\mathrm{F}} = \sum_{i=1}^{N_t} w_{i,t}^F \sigma_{\varepsilon_i,t}^2$; and

$$(A.10) \qquad IV_t^S = A_t^S \sigma_{H,t}^2 - B_t^S \sigma_{MH,t} + \Omega_t^S,$$

where $A_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \beta_{iH,t} b_{iH,t}$, $B_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \beta_{iH,t} b_{iM,t}$, $\Omega_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \sigma_{\varepsilon_i,t}^2$. From (A.9) and (A.10), we can express $\sigma_{H,t}^2$ and $\sigma_{MH,t}$ as linear combinations of $\widetilde{IV}_t^F$ and $\widetilde{IV}_t^S$ defined as:

$$(A.11) \qquad \begin{cases} \widetilde{IV}_t^F \equiv IV_t^F - \Omega_t^F = A_t^F \sigma_{H,t}^2 - B_t^F \sigma_{MH,t} \\ \widetilde{IV}_t^S \equiv IV_t^S - \Omega_t^S = A_t^S \sigma_{H,t}^2 - B_t^S \sigma_{MH,t} \end{cases}.$$

Specificially,

$$(A.12) \qquad \begin{cases} \sigma_{H,t}^2 = \frac{B_t^S}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^F - \frac{B_t^F}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^S \\ \sigma_{MH,t} = \frac{A_t^S}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^F - \frac{A_t^F}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^S \end{cases}.$$

This completes the proof.

C. Proof of Proposition 2

The proposition can be derived from Corollary 1.1 and the ICAPM pricing relationships. Plugging the expression for $\sigma_{MH,t}$ in equation (12) to the right-hand side of equation (4), we obtain the following expression for the conditional equity risk premium:

$$(A.13) \qquad \mu_{M,t} = \gamma_M \sigma_{M,t}^2 + \gamma_H \sigma_{MH,t} = \gamma_M \times \sigma_{M,t}^2 + C_t^F \times \widetilde{IV}_t^F - C_t^S \times \widetilde{IV}_t^S,$$

where $C_t^F = \frac{\gamma_H A_t^S}{A_t^F B_t^S - A_t^S B_t^F}$ and $C_t^S = \frac{\gamma_H A_t^F}{A_t^F B_t^S - A_t^S B_t^F}$.

This completes the proof of the first equality in Proposition 2.

As to the second approximate equality, the proof makes use of the first order Taylor expansion of $ln(x)$:

$$(A.14) \qquad ln(x) = ln(x_0) + \frac{1}{x_0}(x - x_0) + O(x^2),$$

when x is close to x_0. We apply (A.14) to $x = C_t^F \times \widetilde{IV}_t^F$ or $x = C_t^S \times \widetilde{IV}_t^S$ around the following point ψ_0 as x_0:

$$(A.15) \qquad \psi_0 \equiv \frac{E(C_t^F \times \widetilde{IV}_t^F) + E(C_t^S \times \widetilde{IV}_t^S)}{2}.$$

When estimating C_t^F and C_t^S using an expanding window, we find that $C_t^F \times \widetilde{IV}_t^F$ is on average close to $C_t^S \times \widetilde{IV}_t^S$. For example, the average $C_t^F \times \widetilde{IV}_t^F$ over the sample period is -0.0146, while the average $C_t^S \times \widetilde{IV}_t^S$ is -0.0098. Their time-series standard deviations are 0.0061 and 0.0044 respectively. The Taylor expansion gives:

$$(A.16) \qquad \begin{cases} ln\left(C_t^F \times \widetilde{IV}_t^F\right) \approx ln(\psi_0) + \dfrac{1}{\psi_0} C_t^F \times \widetilde{IV}_t^F - 1 \\ ln\left(C_t^S \times \widetilde{IV}_t^S\right) \approx ln(\psi_0) + \dfrac{1}{\psi_0} C_t^S \times \widetilde{IV}_t^S - 1 \end{cases}.$$

Taking the difference between the two equations above:

$$(A.17) \qquad C_t^F \times \widetilde{IV}_t^F - C_t^S \times \widetilde{IV}_t^S \approx \psi_0 \ln\left(\frac{C_t^F}{C_t^S}\right) + \psi_0 \ln\left(\frac{\widetilde{IV}_t^F}{\widetilde{IV}_t^S}\right).$$

Plugging (A.17) into (A.13), we obtain equation (13) stated in Proposition 2. This completes the proof.

D. Proof of Proposition 3

The tail index proposed by Kelly and Jiang (2014) is:

$$(A.18) \qquad \lambda_{t+1}^{Hill} = \frac{1}{K_{t+1}} \sum_{k=1}^{K_{t+1}} ln\left(\frac{\eta_{k,t+1}}{u_{t+1}}\right),$$

where $\eta_{k,t+1}$ is the *k*th daily residual return that falls below an extreme value threshold u_{t+1} during month *t*+1, u_{t+1} is the 5th percentile of the cross-section of individual stock residual returns, and K_{t+1} is the total number of these exceedances within month *t*+1. The residual returns are obtained after removing the exposures individual stock returns to common return factors under a benchmark factor model such as the CAPM. Based on (A.1), (A.3), and Proposition 1, we have:

$$(A.19) \qquad \begin{cases} \eta_{k,t+1} = E_t(\eta_{k,t+1}) + e_{k,t+1} = \gamma_H b_{kH,t}\sigma_{H,t}^2 - \gamma_H b_{kM,t}\sigma_{MH,t} + e_{k,t+1} \\ u_{t+1} = E_t(u_{t+1}) + e_{u,t+1} = \gamma_H b_{uH,t}\sigma_{H,t}^2 - \gamma_H b_{uM,t}\sigma_{MH,t} + e_{u,t+1} \end{cases},$$

where $e_{k,t+1}$ is a random variable with mean zero. The mean of $\bar{\eta}_{t+1} = \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}} \eta_{k,t+1}$ and u_{t+1} are both linear in $\sigma_{H,t}^2$ and $\sigma_{MH,t}$:

$$(A.20) \qquad \begin{cases} E_t(\bar{\eta}_{t+1}) = \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}} E_t(\eta_{k,t+1}) = D_t^{\bar{\eta}}\sigma_{H,t}^2 - G_t^{\bar{\eta}}\sigma_{MH,t} \\ E_t(u_{t+1}) = D_t^u\sigma_{H,t}^2 - G_t^u\sigma_{MH,t} \end{cases},$$

where $\begin{cases} D_t^{\bar{\eta}} = \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}} \gamma_H b_{kH,t} \\ D_t^u = \gamma_H b_{uH,t} \end{cases}$, $\begin{cases} G_t^{\bar{\eta}} = \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}} \gamma_H b_{kM,t} \\ G_t^u = \gamma_H b_{uM,t} \end{cases}$.

From the two equations in (A.20), the covariance risk $\sigma_{MH,t}$ can be identified from $E_t(\bar{\eta}_{t+1})$ and $E_t(u_{t+1})$:

$$(A.21) \qquad \sigma_{MH,t} = J_t^{\bar{\eta}}E_t(\bar{\eta}_{t+1}) - J_t^u E_t(u_{t+1}) = J_t^{\bar{\eta}}\bar{\eta}_{t+1} - J_t^u u_{t+1} + \left(J_t^u e_{u,t+1} - J_t^{\bar{\eta}} e_{\bar{\eta},t+1}\right),$$

where $J_t^{\bar{\eta}} = \frac{D_t^u}{D_t^{\bar{\eta}}G_t^u - D_t^u G_t^{\bar{\eta}}}$ and $J_t^u = \frac{D_t^{\bar{\eta}}}{D_t^{\bar{\eta}}G_t^u - D_t^u G_t^{\bar{\eta}}}$. To link the right-hand side of (A.18) to (A.21), we apply the first order Taylor expansion of $ln(x)$ in (A.14) with $x = J_t^{\bar{\eta}}\bar{\eta}_{t+1}$ or $x = J_t^u u_{t+1}$ around the following point ψ_0 as x_0:

$$(A.22) \qquad \psi_0 \equiv \frac{E\left(J_t^{\bar{\eta}}\bar{\eta}_{t+1}\right) + E(J_t^u u_{t+1})}{2}.$$

When estimating $J_t^{\bar{\eta}}$ and J_t^u using an expanding window, we find that, $J_t^{\bar{\eta}}\bar{\eta}_{t+1}$ is on average close to $J_t^u u_{t+1}$. For example, the average $J_t^{\bar{\eta}}\bar{\eta}_{t+1}$ over the sample period is 0.0328, while the average $J_t^u u_{t+1}$ is 0.0335. Their time-series standard deviations are 0.0320 and 0.0288 respectively. The Taylor expansion gives:

$$(A.23) \qquad \begin{cases} ln(J_t^{\bar{\eta}}\bar{\eta}_{t+1}) \approx ln(\psi_0) + \frac{1}{\psi_0}J_t^{\bar{\eta}}\bar{\eta}_{t+1} - 1 \\ ln(J_t^{u}u_{t+1}) \approx ln(\psi_0) + \frac{1}{\psi_0}J_t^{u}u_{t+1} - 1 \end{cases}.$$

Taking the difference between the two equations above gives:

$$(A.24) \qquad J_t^{\bar{\eta}}\bar{\eta}_{t+1} - J_t^{u}u_{t+1} \approx \psi_0\, ln\left(\frac{J_t^{\bar{\eta}}}{J_t^{u}}\right) + \psi_0\, ln\left(\frac{\bar{\eta}_{t+1}}{u_{t+1}}\right).$$

From (A.21) and (A.24), the conditional covariance risk can be approximated by:

$$(A.25) \qquad \begin{aligned} \sigma_{MH,t} &= J_t^{\bar{\eta}}\bar{\eta}_{t+1} - J_t^{u}u_{t+1} + \left(J_t^{u}e_{u,t+1} - J_t^{\bar{\eta}}e_{\bar{\eta},t+1}\right) \\ &\approx \psi_0\, ln\left(\frac{J_t^{\bar{\eta}}}{J_t^{u}}\right) + \psi_0\, ln\left(\frac{\bar{\eta}_{t+1}}{u_{t+1}}\right) + \left(J_t^{u}e_{u,t+1} - J_t^{\bar{\eta}}e_{\bar{\eta},t+1}\right). \end{aligned}$$

Similar Taylor expansion gives:

$$(A.26) \qquad ln\left(\frac{\eta_{k,t+1}}{u_{t+1}}\right) \approx ln(\phi_{0,t}) + \frac{1}{\phi_{0,t}}\left(\frac{\eta_{k,t+1}}{u_{t+1}} - \phi_{0,t}\right),$$

where $\phi_{0,t} = E_t\left(\frac{\eta_{k,t+1}}{u_{t+1}}\right)$. Thus, the tail index of Kelly and Jiang (2014) can be approximated as:

$$(A.27) \qquad \begin{aligned} \lambda_{t+1}^{Hill} &\approx \frac{1}{K_{t+1}}\sum_{k=1}^{K_{t+1}}\left(ln\left(\phi_{0,t}\right) + \frac{1}{\phi_{0,t}}\left(\frac{\eta_{k,t+1}}{u_{t+1}} - \phi_{0,t}\right)\right) \\ &= ln\left(\phi_{0,t}\right) + \frac{1}{\phi_{0,t}}\left(\frac{\overline{\eta}_{t+1}}{u_{t+1}} - \phi_{0,t}\right) \\ &\approx ln\left(\frac{\overline{\eta}_{t+1}}{u_{t+1}}\right). \end{aligned}$$

(A.25) and (A.27) lead to the following relationship between the conditional covariance $\sigma_{MH,t}$ and the tail index of Kelly and Jiang (2014):

$$(A.28) \qquad \sigma_{MH,t} \approx \psi_0 \, ln\left(\frac{J_t^{\overline{\eta}}}{J_t^u}\right) + \psi_0 \, ln\left(\frac{\overline{\eta}_{t+1}}{u_{t+1}}\right) + \left(J_t^u e_{u,t+1} - J_t^{\overline{\eta}} e_{\overline{\eta},t+1}\right)$$

$$\approx \psi_0 \, ln\left(\frac{J_t^{\overline{\eta}}}{J_t^u}\right) + \psi_0 \lambda_{t+1}^{Hill} + \left(J_t^u e_{u,t+1} - J_t^{\overline{\eta}} e_{\overline{\eta},t+1}\right).$$

This is equivalent to Proposition 3:

$$(A.29) \qquad \lambda_{t+1}^{Hill} \approx ln\left(\frac{J_t^u}{J_t^{\overline{\eta}}}\right) + \frac{1}{\psi_0} \sigma_{MH,t} + e_{t+1},$$

where e_{t+1} has mean zero. Thus, the tail index of Kelly and Jiang (2014) is proportional to the conditional covariance $\sigma_{MH,t}$ under the ICAPM. This completes the proof.