# Trading in Crowded Markets

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**Abstract**

We study trading among strategic traders who may incorrectly assess the degree of market crowdedness. These mistakes distort equilibrium strategies and prices. When traders underestimate market crowdedness, they target larger inventories and trade more aggressively, but their actual profits are lower than expected because they underestimate the amount of information already impounded into prices. Crowded markets are prone to abrupt crashes. The magnitude of price dislocations and the speed of recovery during fire-sale events can help infer traders' beliefs about market crowdedness.

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# I. Introduction

With the dramatic growth of the asset management industry, financial markets have become a playing field where sophisticated institutional players trade intensively with one another. Even though asset managers strive to differentiate their trading strategies from others, they often build their portfolios on correlated signals, and this leads to crowded markets. Investors and regulators are concerned about crowded markets because of their perceived fragility.

In his 2009 Presidential address to the American Finance Association, Jeremy Stein brings the "crowded-trade" problem to the attention of researchers. He notes that "for a broad class of quantitative trading strategies, an important consideration for each individual arbitrageur is that he cannot know in real time exactly how many others are using the same model and taking the same position as him" (Stein (2009), p. 1518).

Because market crowdedness is difficult to observe in real time, traders may submit orders based on misestimated parameters of market crowdedness. Stein (2009) illustrates the consequences of these errors by discussing market events during the change in the composition of MSCI indices in 2001–2002. In anticipation of trading by index managers, arbitrageurs bought stocks whose weights were known to increase and sold stocks whose weights were known to decrease. Yet this strategy did not yield gains as expected, because too many arbitrageurs rushed to take advantage of the opportunity and prices overreacted.

Recently, crowded trading has become an important subject of public policy discussion. Regulators are increasingly concerned about whether some strategies and market segments may have become crowded and may be at the risk of unwinding (e.g., Banque de France (2007)). Menkveld (2017) warns that crowded trades and concentration on a small set of risk factors may create a systemic risk for a central clearing party. Basak and Pavlova (2013) discuss how institutional investors may make markets crowded and amplify market fragility. Brown, Howard, and Lundblad (2022) empirically document that crowdedness tends to exacerbate price crashes during periods of distress. Kashyap, Kovrijnykh, Li, and Pavlova (2023) suggest that crowded trades can arise due to the prevalence of benchmarking. To provide tools for assessing crowdedness of strategies, the Morgan Stanley Capital International (MSCI) introduced its "crowding scorecard."

In this paper, we formally examine the impact of traders' incorrect beliefs about market crowdedness on equilibrium strategies, price dynamics, trading profits, and market liquidity. We consider market crowdedness to be related to the average correlations of private signals. High correlations of signals can stem from two sources: the correlation of noise terms and the correlation in the information content. These two cases have different implications. If high correlation results from the correlation of noise terms in signals, traders are more willing to provide liquidity to each other and markets are more liquid. In contrast, if high correlation arises from the information content of signals, traders trade less aggressively and markets are less liquid. We focus on the latter case—high correlation in the information content of signals—because we are interested in modeling crowdedness among sophisticated institutional investors who tend to trade in the same direction based on similar fundamental information and take into account the impact of their trading on prices. In particular, we analyze the effects of market crowdedness by varying the proportion of traders with high-precision signals.

A high perceived correlation in the information content of signals suggests that traders believe they are observing similar signals, leading them to trade in the same direction at the same time. However, market clearing prevents this from happening, causing market prices to move in the direction of traders' trades. This captures how sophisticated institutional investors typically think of crowded markets (e.g., Banque de France (2007) and Arnott, Harvey, Kalesnik, and Linnainmaa (2019)).

Our one-period model generates the following comparative statics results. When traders adopt a new trading strategy, they usually don't know in real time exactly how many others are following a similar strategy. If traders underestimate market crowdedness and the signals are more correlated than they believe, they tend to trade more intensively, take larger positions, and supply more liquidity to others. Traders expect their profits to be high, but the actual profits are lower than expected, because traders underestimate the amount of information already impounded into prices. This is consistent with the losses experienced by arbitrageurs trading on changes in the composition of MSCI indices in Jeremy Stein's example.

On the other hand, if traders begin to overestimate market crowdedness as a trading strategy becomes more popular, they are less willing to provide liquidity to each other, target smaller inventory levels, and trade less aggressively. The market becomes less liquid. Traders expect lower profits, but the actual profits may turn out to be higher than expected.

To study the dynamic properties of crowded markets, we extend our static model to a dynamic model in which traders "shred orders" and trade gradually toward target inventories. We show that when traders believe that markets are more crowded, they trade more slowly towards their targets, thus making actual inventories deviate further away from target levels. Disagreement about the precisions of private signals leads to positive autocorrelation (time-series momentum) in returns. When traders underestimate market crowdedness, they trade more aggressively on short-run profit opportunities and this makes time-series return momentum more pronounced. This is in line with the empirical findings of Lee and Swaminathan (2000) and Cremers and

Pareek (2014) that time-series momentum is more pronounced for stocks with higher volume and more short-term trading.

Our paper contributes to the theoretical literature on crowded markets. Stein (2009) suggests that crowding among arbitrageurs may cause prices to be pushed further away from fundamentals. Callahan (2004) analyzes the model of Kyle (1985) when the total number of informed traders is some unknown number equal to either one or two. Banerjee and Green (2015) and Papadimitriou (2023) find that the interaction between learning about fundamentals and the market composition makes prices react more strongly to negative news than to positive news. Easley, O'Hara, and Yang (2014) show that ambiguity aversion of mutual funds about the trading strategies of hedge funds may limit their ability to infer information from prices and thus have negative effects on market outcome. Kondor and Zawadowski (2019) find that additional entrants do not improve the efficiency of capital allocation and decrease social welfare. Extending this literature, we study how traders' mistakes about market crowdedness may distort equilibrium strategies and prices.

Our model is based on the smooth trading framework of Kyle, Obizhaeva, and Wang (2018), but we extend it to focus on a different set of questions. We assume that multiple traders, rather than a single trader as in the original model, may observe private information with high precision. We also allow private signals to be correlated. These modifications enable us to study the effects of crowdedness among sophisticated institutional investors and, in particular, how their incorrect beliefs about this parameter affect market characteristics.

Our paper also relates to the empirical literature on crowded markets. Pojarliev and Levich (2011) measure style crowdedness in currency trades as the percentage of funds with significant positive exposure minus the percentage of funds with significant negative exposure to a given style. Polk and Lou (2022) gauge the level of arbitrageurs' crowdedness in momentum strategies from daily and weekly abnormal return correlations among stocks in the winner or loser portfolios. Sokolovski (2018) applies both measures to analyze dynamics in returns of carry trades. Hong, Li, Ni, Scheinkman, and Yan (2016) use days-to-cover metrics, defined as the ratio of a stock's short interest to trading volume to approximate the cost of exiting crowded trades. Yan (2013) combines the short interest ratio and the exit rate of institutional investors, defined as the number of shares liquidated, to study momentum strategies. Consistent with our results, researchers usually find that strategies work well as long as they are not crowded, and they tend to crash or revert when crowdedness increases. As in our model, empirical measures of crowdedness are related to the number of traders and correlations in their signals.

Our model offers several additional insights about possible empirical properties of crowded markets. Elevated levels of crowdedness are likely associated with a combination of high levels of institutional ownership, low rates of share turnover, and low market liquidity. This is in line with the main measure of market crowdedness used by Brown et al. (2022), who define the crowdedness of a stock as the percentage of shares owned by hedge funds relative to the stock's average daily volume (i.e., the number of days it would take hedge funds to exit their collective position). Consistent with our model's predictions, they also find that during periods of industry distress, crowded stocks experience larger price declines followed by slower recoveries.

In addition, we propose a potential approach to infer market crowdedness. We show that in more crowded markets, flash crashes tend to be more severe, and subsequent price recoveries are more prolonged. This suggests that the level of market crowdedness, along with traders' perceptions of this parameter, can be inferred by analyzing flash crash events in the context of our dynamic model and calibrating the model's parameters to the observed price drop and the speed of price recovery. To illustrate, we show how to identify traders' misestimation of market crowdedness during the "quant meltdown" episode in August of 2007, when several hedge funds following similar strategies experienced massive losses (see Khandani and Lo (2011) and Pedersen (2009)). Similar analysis can be applied to assess market crowdedness during episodes when certain market participants rapidly liquidate substantial positions. For example, anecdotal evidence suggests that market crowdedness may have played a key role during the unwinding of carry trades and during momentum crashes. Our paper indicates that analyzing these events may shed light on how market participants assess market crowdedness.

Our analysis suggests that it is important for regulators and market participants not only to monitor the risks of major investment strategies becoming overcrowded but also to track the evolution of the market's beliefs about relevant characteristics. By doing so, they can better understand the risks of market dislocations and help ensure the stability of financial markets. Since, as shown in our calibration exercise, even small misestimations can lead to significant price drops, improved disclosure of asset managers' positions would be beneficial. A better understanding of current levels of market crowdedness can help prevent destabilizing market movements.

The paper is structured as follows. Section II presents a one-period model of crowded markets. Section III examines dynamic properties of prices and trading strategies in crowded markets. Section IV discusses how to infer traders' misestimation of market crowdedness. Section V concludes. All proofs are in the Appendix.

# II. A Static Model

## A. Model Setup

We begin with a one-period model where traders might hold incorrect beliefs regarding specific parameters, while understanding the overall model setting. We next discuss equilibrium strategies based on these potentially incorrect beliefs. In Section E, we use the true model parameters to derive the actual distribution of profits.

There is one risky asset traded against a safe asset. Each trader believes that there are $N$ strategic traders trading the risky asset, which has a random payoff of $v$, $v \sim N(0, 1/\tau_v)$. Each trader $n$ is endowed with inventory $S_n$ of the risky asset. It is common knowledge that the asset is in zero net supply, and thus $\sum_{n=1}^{N} S_n = 0$.

Each trader $n$ observes a private signal $i_n$ with a perceived precision $\tau_n$ and a noise term. This noise term correlates with noise terms in other traders' private signals with a perceived correlation coefficient of $\rho$,

(1) $$i_n := \tau_n^{1/2} \left(\tau_v^{1/2} v\right) + \rho^{1/2} z + (1-\rho)^{1/2} e_n,$$

where $z \sim N(0,1)$, $e_n \sim N(0,1)$, $n = 1, 2, \ldots, N$, and $v$, $z$, $e_1$, …, $e_N$ are independently distributed.[1] In the symmetric equilibrium, each trader believes he infers from prices the average of other traders' signals, denoted as $i_{-n}$,

(2) $$i_{-n} := \frac{1}{N-1} \sum_{m=1, m \neq n}^{N} i_m.$$

To motivate speculative trading among strategic investors in the symmetric model with no

[1] For expositional simplicity, we assume there are no public signals in the one-period model. Including public signals does not change any of our main results.

noise traders, we assume that traders symmetrically agree to disagree about the precision of private signals. Traders believe that there are $N_H$ traders with private signals of high precision $\tau_H$ and $N_L$ traders with private signals of low precision $\tau_L$, with $\tau_H > \tau_L \geq 0$ and $N_L + N_H = N$.

Each trader believes that his own signal has high precision $\tau_H$,[2] and that the fraction of other traders (except trader $n$ himself) who observe signals of high precision is given by

$$(3) \qquad \theta := \frac{N_H - 1}{N_H + N_L - 1}.$$

Traders may hold incorrect beliefs about this fraction. The true (objective) fraction of traders who receive high-precision signals is denoted by $\hat{\theta}$, which may differ from $\theta$, as discussed in Section E.

Each trader believes that the average of others' signals is

$$(4) \qquad i_{-n} = (\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2})\,(\tau_v^{1/2} v) + \rho^{1/2} z + (1-\rho)^{1/2} \tfrac{1}{N-1} \sum_{m=1, m\neq n}^{N} e_m.$$

This average of others' signals can be inferred from the price.

Upon observing signals $i_n$, and given his current inventory $S_n$, each trader $n$ submits a demand schedule $X_n(P) := X_n(i_n, S_n, P)$ as a function of price $P$ to a single-price auctioneer who clears the market at price $P$. Each trader $n$ chooses his optimal demand to maximize the expected negative exponential utility function with risk aversion $A$,

$$(5) \qquad \max_{X_n(P)} \mathrm{E}^n[-\mathrm{e}^{-A\,W_n}],$$

where $\mathrm{E}^n$ denotes that trader $n$ uses his own beliefs to calculate the expectation. Trader $n$'s

[2] Agreement to disagree about the precision of private signals is a modeling device to generate speculative trading among strategic informed traders. If $\tau_H = \tau_L$, each trader's target inventory is always zero, meaning they trade solely to offload their initial inventory if the market is competitive. In the imperfectly competitive model, as explained below after Theorem 1, when each trader believes that others may also observe informative private signals ($\theta \neq 0$ or $\tau_L \neq 0$), each trader is subject to adverse selection risk. In the presence of adverse selection, an equilibrium with positive trading volume does not exist if there is no noise demand, no endowment shocks (hedging demand), and no disagreement.

terminal wealth is given as $W_n := v\ (S_n + X_n(P)) - P\ X_n(P)$. Each trader $n$ strategically exercises market power by taking into account that his trading quantity $x_n$ directly affects the price $P(x_n)$ on the residual supply schedule he believes to be facing.

## B. Discussions on the Assumptions

Since we focus on crowded trading by large asset managers who exploit private information about securities, we adopt the current setting in which traders agree to disagree about the precision of private signals and speculate on their private information about fundamentals. Specifically, each trader believes that he observes a signal of higher precision than others ($\tau_H > \tau_L$).

We present an alternative model without disagreement in Section B.1 of the Online Appendix. In this alternative setting, we replace disagreement about signal precisions (different priors) with private values (a common prior). We show that as the correlation of private values increases, the correlation of traders' composite signals also rises. Traders provide more (not less) liquidity to each other, leading to a more liquid market and higher ex-ante expected trading profits. Similar results are obtained in a model where traders share a common prior about the precision of private signals but trade due to endowment shocks. As the correlation of endowment shocks (which are unrelated to fundamentals) increases, traders become more willing to provide liquidity to each other, markets become more liquid, and ex-ante expected trading profits rise.

Since our goal is to model crowdedness among sophisticated traders, we focus on studying the effects of market crowdedness by varying the correlation in the information content of signals, specifically by varying the fraction of traders with high-precision signals ($\theta$). A high correlation in the information content of signals suggests that traders believe they are observing similar signals. This leads them to trade in the same direction at the same time, affecting prices and reducing trading profitability. This result reflects how sophisticated institutional investors typically perceive crowded markets (e.g., Banque de France (2007) and Arnott et al. (2019)) and aligns with empirical findings that trading strategies perform well when they are not crowded but tend to revert as crowdedness increases (e.g., Pojarliev and Levich (2011), Polk and Lou (2022), Sokolovski (2018), Hong et al. (2016), and Yan (2013)).

The assumption of disagreement about signal's precisions and possibly incorrect beliefs about the fraction of traders who observe high-precision signals form a coherent set of assumptions. For example, we can assume that each trader $n$ believes that $N_H$ traders (including trader $n$) in the market observe an informative signal ($\tau_H > 0$) while the remaining traders observe pure noise ($\tau_L = 0$). If each trader $n$ believes that he observes pure noise, then he will not participate. In a market with active institutional traders trading on their private signals, each trader cannot know in real time exactly how many others are trading based on similar information in the same direction. Therefore, it is likely that they might have incorrect beliefs about the fraction of traders who observe high-precision signals.

## C. Model Solution

We describe the unique equilibrium with linear trading strategies. Each trader $n$ observes his own private signal $i_n$, and infers a perceived average signal of others $i_{-n}$ from the equilibrium price. Based on this information, he updates his estimates of the expectation and the variance of fundamental value $v$, denoted $\mathrm{E}^n[v]$ and $\mathrm{Var}^n[v]$, respectively. Define "perceived total precision" $\tau$ as the inverse of posterior variance $\tau := (\mathrm{Var}^n[v])^{-1}$. The projection theorem for jointly normally distributed random variables yields the following result:

**Lemma 1.** *Each trader n forms the following estimates of posterior expectation and variance:*

$$\tau := (\mathrm{Var}^n[v])^{-1} = \tau_v\Big(1 + \tau_H + (N-1)\frac{\big((\theta-\rho)\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}\big)^2}{(1-\rho)(1+(N-1)\rho)}\Big), \tag{6}$$

$$\mathrm{E}^n[v] = \frac{\tau_v^{1/2}}{\tau}\left(\frac{1-\theta}{1-\rho}\left(\tau_H^{1/2} - \tau_L^{1/2}\right) i_n + \frac{(\theta-\rho)\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}\ \big(i_n + (N-1)i_{-n}\big)\right). \tag{7}$$

Trader $n$'s terminal wealth $W_n$ is a normally distributed random variable. Maximizing the expected utility function (5) is equivalent to maximizing $\mathrm{E}^n[W_n] - \frac{1}{2}A\,\mathrm{Var}^n[W_n]$, where the mean and variance are given by

$$\mathrm{E}^n[W_n] = \mathrm{E}^n[v]\,(S_n + x_n) - P(x_n)\,x_n, \qquad \mathrm{Var}^n[W_n] = (S_n + x_n)^2\,\mathrm{Var}^n[v]. \tag{8}$$

Conjecturing that other traders submit symmetric linear demand schedules,

$$X_m(i_m, S_m, P) = \gamma_I\, i_m - \gamma_P\, P - \gamma_S\, S_m, \qquad m = 1, \ldots, N, \quad m \neq n, \tag{9}$$

and believing the market-clearing condition holds in equilibrium,

$$x_n + \sum_{\substack{m=1 \\ m\neq n}}^{N} (\gamma_I\, i_m - \gamma_P\, P - \gamma_S\, S_m) = 0, \tag{10}$$

each trader $n$ infers a residual supply schedule $P(x_n)$ that he believes to be facing as a function of quantity $x_n$ he trades:

$$P(x_n) = \frac{\gamma_I}{\gamma_P}\, i_{-n} + \frac{\gamma_S}{(N-1)\gamma_P}\, S_n + \frac{1}{(N-1)\gamma_P}\, x_n. \tag{11}$$

The more trader $n$ buys, the higher the price he has to pay. The more he sells, the lower the price he has to accept.

The inverse slopes of this function with respect to $S_n$ and $x_n$ are two measures of perceived market depth $1/\lambda$ and $1/\kappa$, respectively:

$$\frac{1}{\lambda} := \frac{(N-1)\,\gamma_P}{\gamma_S}, \qquad \frac{1}{\kappa} := (N-1)\,\gamma_P. \tag{12}$$

In the continuous-time model presented in Section III, $\lambda$ is interpreted as the permanent price impact and $\kappa$ is interpreted as the temporary price impact; the permanent price impact depends on the size of executed orders ($S_n$), and the temporary price impact depends on the execution speed ($x_n$).

We next plug $\mathrm{E}^n[W_n]$ and $\mathrm{Var}^n[W_n]$ from equation (8), and $P(x_n)$ from equation (11) into the utility maximization and solve for equilibrium quantities and prices. The following theorem characterizes the equilibrium.

**Theorem 1. One-Period Model with Imperfect Competition.** *An equilibrium with linear trading strategies exists if and only if the second-order condition holds:*

$$\theta < 1 - \left(\tfrac{N}{N-1}\right)\left(\tfrac{1-\rho}{2+(N-2)\rho}\right)\left(\frac{\tau_H^{1/2}}{\tau_H^{1/2}-\tau_L^{1/2}}\right) < 1. \tag{13}$$

*If an equilibrium with linear trading strategies exists, then:*

1. *The parameters $\gamma_S$, $\gamma_I$, $\gamma_P$ defining the optimal trading strategy have unique closed-form solutions with $0 < \gamma_S < 1$, $\gamma_I > 0$, and $\gamma_P > 0$:*

$$\gamma_S = 1 - \tfrac{1-\rho}{1+(N-1)\rho}\left(\tfrac{N}{N-1}\,\frac{\tau_H^{1/2}}{(1-\theta)(\tau_H^{1/2}-\tau_L^{1/2})} - 1\right), \qquad \gamma_I = \frac{(1-\theta)(\tau_H^{1/2}-\tau_L^{1/2})}{A(1-\rho)}\,\tau_v^{1/2}\,\gamma_S,$$

$$\gamma_P = \frac{\tau(1+(N-1)\rho)}{(1+(N-1)\theta)\tau_H^{1/2}+(N-1)(1-\theta)\tau_L^{1/2}}\;\frac{\gamma_I}{\tau_v^{1/2}}. \tag{14}$$

2. *Trader $n$ trades the quantity $x_n$ towards his target inventory $S_n^{TI}$:*

$$S_n^{TI} = \frac{1}{A}\,\frac{1-\theta}{1-\rho}\left(1-\tfrac{1}{N}\right)\tau_v^{1/2}\,(\tau_H^{1/2}-\tau_L^{1/2})\,(i_n - i_{-n}), \qquad x_n = \gamma_S\,(S_n^{TI} - S_n), \tag{15}$$

*the equilibrium strategies (15), depend on the subjective parameter values $N$, $\theta$, $\rho$, $\tau_H$, $\tau_L$, and $\tau_v$ rather than on the objective parameter values, as will be discussed in Section E.*

3. *Traders believe that the price $P$ is the average valuation of traders:*

$$P = \tfrac{1}{N}\sum_{n=1}^{N} \mathrm{E}^n[v] = \frac{\gamma_I}{\gamma_P}\,\frac{i_n + (N-1)\,i_{-n}}{N}. \tag{16}$$

*If the perceived number of traders $N$ is correct, this perceived price corresponds to the actual price.*

If $\tau_L = 0$ and $\theta = 0$, each trader believes that he is the only one observing informative signals, while others observe pure noise and trade based on this noise. As a result, each trader believes that he is not subject to adverse selection risk. In this case, the existence condition (13) is satisfied as long as $N \geq 3$.

When each trader believes that others might also observe informative private signals ($\theta \neq 0$ or $\tau_L \neq 0$), each trader is subject to adverse selection risk. Equation (13) implies that, for an equilibrium with positive trading volume to exist, traders must believe that the fraction $\theta$ of traders with signals of high precision must be not too high.[3] Intuitively, holding other parameters fixed, if the fraction of traders with high precision signals ($\theta$) is too large, the adverse selection risk becomes too high. Alternatively, for any given $\theta \geq 0$, if the disagreement among traders is not

[3] The existence condition is reduced to $\tau_H^{1/2}/\tau_L^{1/2} > 2 + \frac{2}{N-2}$ if $\rho = 0$ and $\theta = 0$, as in the setting of Kyle et al. (2018). This condition requires $N \geq 3$ and $\tau_H^{1/2}$ to be sufficiently more than twice as large as $\tau_L^{1/2}$ for equilibrium to exist.

sufficiently large, i.e., $\frac{\tau_L^{1/2}}{\tau_H^{1/2}} > 1 - \frac{N(1-\rho)}{(1-\theta)(N-1)(2+(N-2)\rho)}$, the benefit of trading is not significant enough. Due to adverse price impact costs, there is no equilibrium with positive trading volume.

To study how misestimates of crowdedness affect traders' aggressiveness, we focus on a setting with imperfect competition. Theorem 1 implies that each imperfectly competitive trader sets a target inventory level, $S_n^{TI}$, and partially adjusts their current inventory, $S_n$, toward this target by a fraction $\gamma_S$, where $0 < \gamma_S < 1$, as shown in equation (14).

Following Lee and Kyle (2018), we interpret the trading aggressiveness parameter $\gamma_S$ in equation (15) as a measure of market competitiveness because it can be shown that $\gamma_S$ is equal to the ratio of equilibrium quantities in the model with imperfect competition to that in an analogous model with perfect competition. In the competitive equilibrium, the trading aggressiveness $\gamma_S$ equals 1. The less competitive the market is, the less aggressively traders trade, and the lower this ratio becomes.

## D. Model Implications When Traders Have Correct Beliefs

We study the effects of market crowdedness by varying the proportion of traders with high-precision signals. As this proportion increases, the correlation between private signals about fundamental values also rises. We first discuss the implications of the model, assuming that traders have correct beliefs about market crowdedness.

The perceived correlation between each trader's private signal and the average of other traders' private signals, based on traders' beliefs, is given as:

$$\text{(17)} \qquad \text{Corr}^n[i_n, i_{-n}] = \frac{\tau_H^{1/2}\left(\tau_L^{1/2}+\theta\left(\tau_H^{1/2}-\tau_L^{1/2}\right)\right)+\rho}{(\tau_H+1)^{1/2}\left(\left(\tau_L^{1/2}+\theta\left(\tau_H^{1/2}-\tau_L^{1/2}\right)\right)^2+\rho+(1-\rho)/(N-1)\right)^{1/2}}.$$

A high perceived correlation in the information content of signals implies that traders believe they observe similar signals. As a result, they often lean toward trading in the same direction at the same time. Of course, market clearing prevents this from happening, so market prices are inclined to move in the direction of trader $n$'s signals. Our model is structured to capture this intuition, which aligns with how sophisticated institutional investors typically perceive crowded markets.[4]

As shown in equation (17), the perceived correlation, $\mathrm{Corr}^n[i_n, i_{-n}]$, depends on $\theta$, $\rho$, $\tau_H$, $\tau_L$, and $N$. While we can alter the perceived correlation by adjusting any of these parameters, we primarily focus on varying the perceived percentage of high-precision traders $\theta$ (holding $N$ constant, therefore increasing $N_H$ and decreasing $N_L$ equivalently).[5] This approach aligns with the observation of Stein (2009) that market participants might have incorrect subjective beliefs about the number of traders who trade in the same direction, especially the more informed ones.

Comparative statics properties of trading strategies depend on perceived parameter values, not objective parameter values. The following proposition describes how market characteristics change when the perceived fraction of traders receiving signals of high precision increases:

**Proposition 1. Comparative Statics Results of Increasing $\boldsymbol{\theta}$.** *If traders believe that the fraction of traders with high-precision signals $\theta$ goes up (increasing $N_H$ and holding $N$ constant), then:*

1. *Trading aggressiveness, $\gamma_S$, which also measures market competitiveness, decreases.*

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[4]The correlation between trader $n$'s signal and the equilibrium price $P$, $\mathrm{Corr}^n[i_n, P]$ is analytically proven to increase monotonically with $\theta$. We also calculate the correlation between trader $n$'s trade and the total trading of the remaining $N_H - 1$ traders who also observe high-precision signals. This correlation initially increases with the fraction of traders who observe high-precision signals, but after a certain point, it starts to decrease. This decrease is due to the market-clearing condition, as there are fewer traders observing low-precision signals who tend to trade in the opposite direction. As expected, trader $n$'s trade is negatively correlated with those traders who observe low-precision signals.

[5]Holding fixed other parameters, direct calculation shows that $\mathrm{Corr}^n[i_n, i_{-n}]$ monotonically increases in $N_H$ and $\theta$. It also monotonically increases in $\tau_L$, but not monotonic in $\tau_H$, $N_L$, or $\rho$.

2. *The willingness to provide liquidity ($\gamma_P$) and the willingness to demand liquidity ($\gamma_I$) both decrease.*

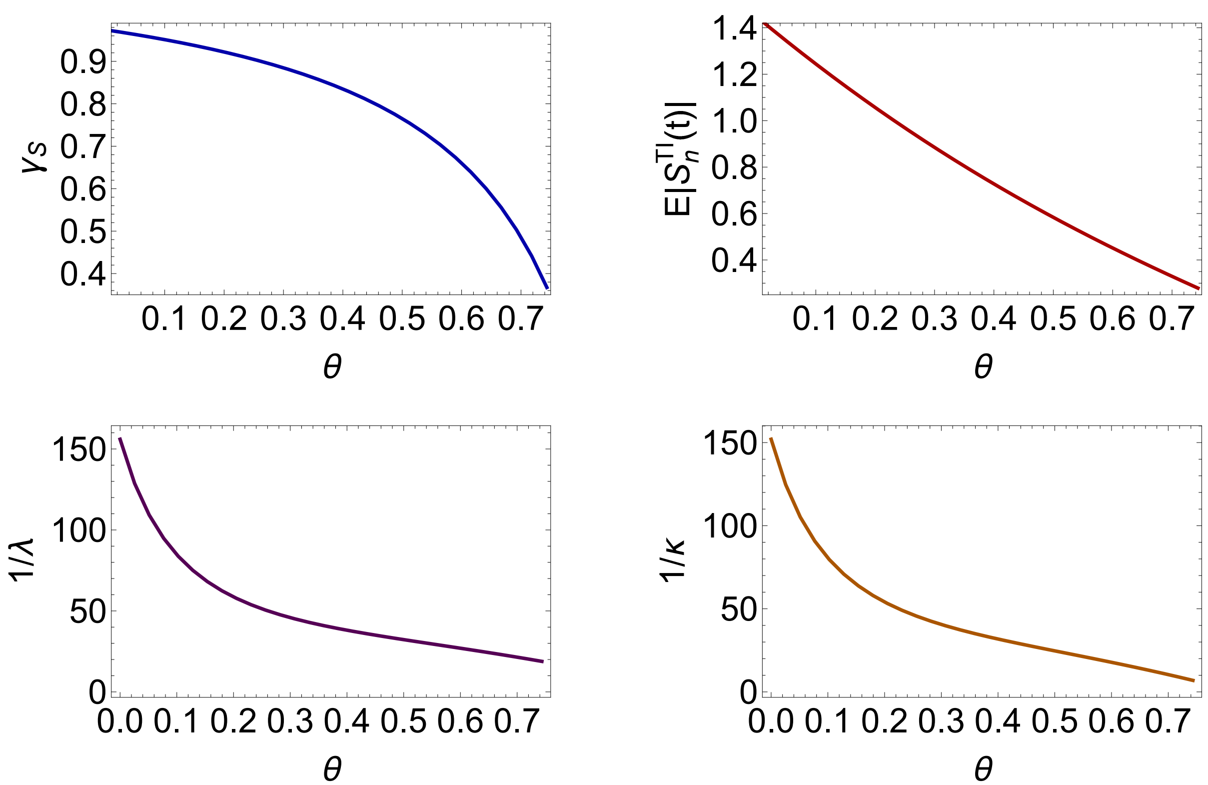
3. *A trader's average target inventory level* $\mathrm{E}^n[|S_n^{TI}|]$ *decreases.*

4. *Perceived market depths* $1/\lambda$ *and* $1/\kappa$ *decrease.*

5. *Traders' ex-ante expected trading profits, denoted as* $\mathrm{E}^n[Profit] := \mathrm{E}^n[(\mathrm{E}^n[v] - P)x_n]$, *decrease when initial inventories are zero ($S_n = 0$).*

FIGURE 1

**Values of $\gamma_S$, $\mathrm{E}|S_n^{TI}(t)|$, $1/\lambda$, and $1/\kappa$ against $\theta$ by varying $N_H$ while keeping $N$ fixed**




The results of Proposition 1 are quite intuitive and consistent with perceived properties of crowded markets. As the proportion of traders with high-precision increases, the perceived correlation of signals rises, each trader becomes less willing to provide liquidity to others, the level of market competitiveness decreases, traders' target inventories decrease, perceived liquidity decreases, and traders trade less aggressively. Each trader, believing himself to be highly informed, has lower perceived expected profits.

FIGURE 2

**Values of $\gamma_S$, $\mathrm{E}|S_n^{TI}(t)|$, $1/\lambda$, and $1/\kappa$ against $\rho$**

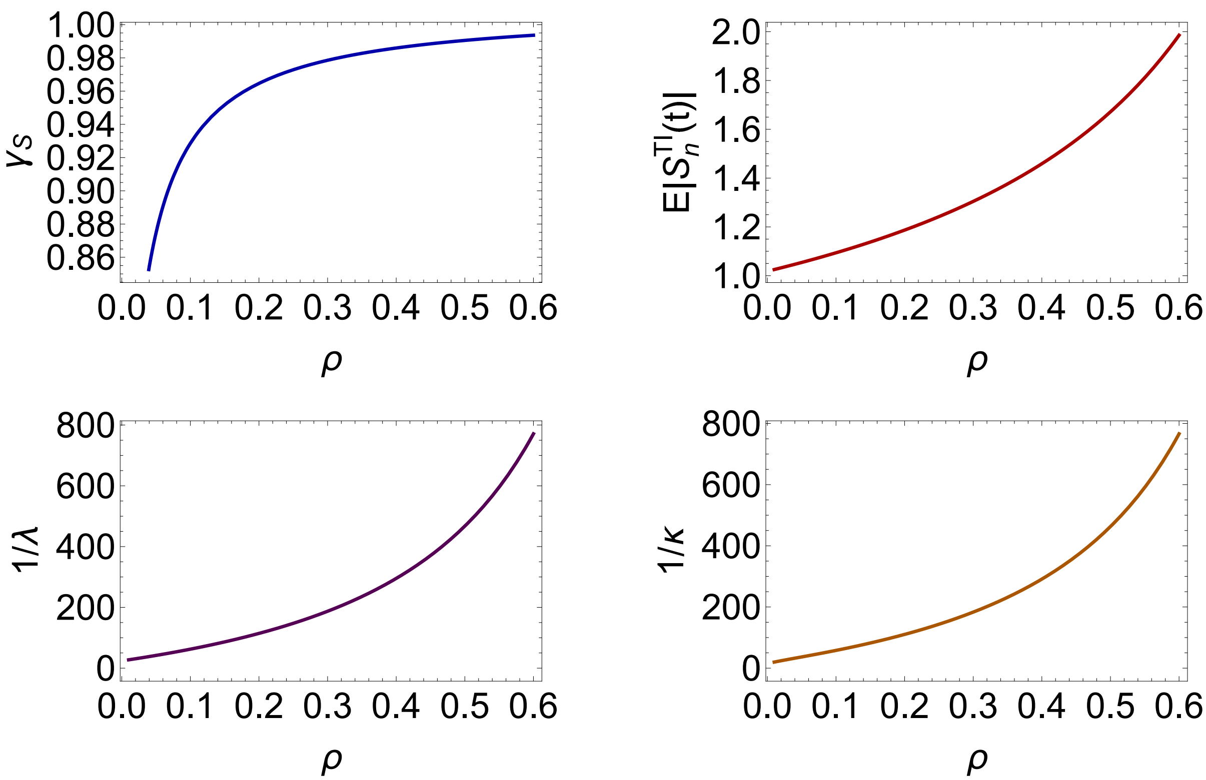



Figure 1 presents the trading aggressiveness $\gamma_S$, the expected size of target positions $\mathrm{E}|S_n^{TI}(t)|$, and the market depths $1/\lambda$ and $1/\kappa$ against the fraction of traders with high-precision signals ($\theta$).[6] When traders perceive a high fraction of traders with high-precision signals, they trade less aggressively and aim for smaller target positions, resulting in a less liquid market.

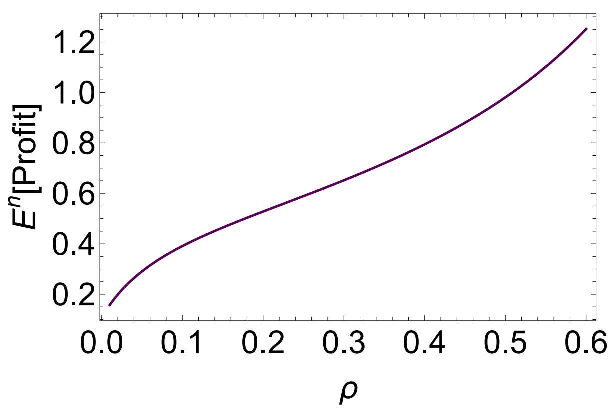
[6] In Figures 1, 2, and 3, the default parameter values are $A = 1$, $\tau_v = 2$, $N = 40$, $N_I = 8$, $\rho = 0.1$, $\tau_H = 1$, and $\tau_L = 0.01$.

Conversely, a lower perceived fraction of traders with high-precision signals leads to more aggressive trading and more liquid markets.

It is worth exploring how market characteristics change when we vary the correlation in the noise terms of private signals, $\rho$. We can analytically show that market depths $1/\lambda$ and $1/\kappa$, trading aggressiveness $\gamma_S$, and the expected size of target positions $\mathrm{E}|S_n^{TI}(t)|$ all increase with the correlation in the noise terms of signals $\rho$, as illustrated in Figure 2. When the correlation in the noise terms (unrelated to fundamentals) of private signals increases, traders become more willing to provide liquidity to one another, leading to more aggressive trading and a more liquid market. Traders' ex-ante expected trading profits also increase with the correlation in the noise terms of signals $\rho$, as illustrated in Figure 3.

FIGURE 3

**Values of $\mathbf{E}^n[\mathbf{Profit}]$ against $\boldsymbol{\rho}$**




This is the opposite of what occurs when we consider changes in the correlation in the information content of private signals. Therefore, it is crucial to distinguish whether the increase in signal's correlation is due to noise or information content, as these different scenarios lead to drastically different implications.

Since our goal is to model crowdedness among sophisticated traders, we focus on the effects of market crowdedness by varying the correlation in the information content of private signals, specifically by varying the fraction of traders with high-precision signals $\theta$. When traders observe similar signals about fundamentals, they tend to trade in the same direction, which affects prices and makes trading less profitable. This is consistent with how traders typically think about crowded markets.

## E. Model Implications When Traders Have Incorrect Beliefs

We next discuss the implications of the model when traders may have incorrect beliefs about market crowdedness. We place a "hat" over a parameter to denote its objective value when this value is potentially different from the subjective value. Let $\hat{\theta}$ denote the true (objective) fraction of traders receiving high-precision signals, which may differ from the perceived fraction $\theta$. Next, we consider its effects on perceived and actual trading profits.

Let $\mathrm{E}[\text{Profit}]$ and $\mathrm{E}^n[\text{Profit}]$ denote the objective and perceived ex-ante expected trading profits for a trader who observes high precision signals under objective beliefs, assuming $S_n = 0$.

**Proposition 2. Perceived and Actual Trading Profits of A Trader with High Precision Signals.** *Assume traders have correct beliefs about the parameters $\rho$, $N$, $\tau_H$, $\tau_L$, $\tau_v$. If traders underestimate market crowdedness ($\theta < \hat{\theta}$), then $\mathrm{E}^n[Profit] > \mathrm{E}[Profit]$. If traders overestimate market crowdedness ($\theta > \hat{\theta}$), then $\mathrm{E}^n[Profit] < \mathrm{E}[Profit]$.*

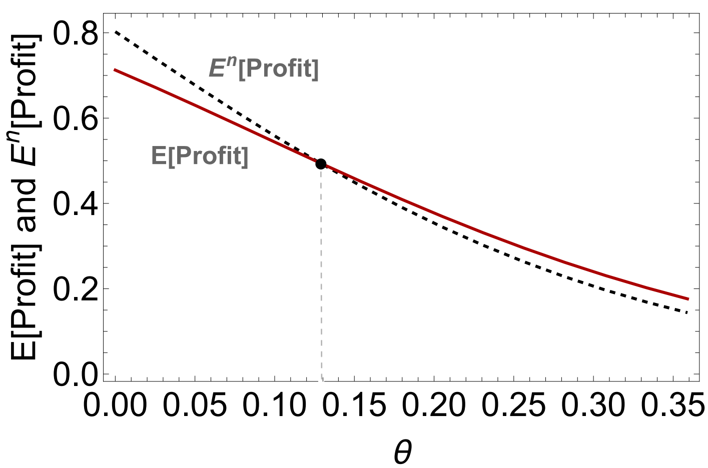
Figure 4 depicts the expected profits of a trader who observes high precision under traders' beliefs $\mathrm{E}^n[\text{Profit}]$ (dashed curve) and the expected profits under true beliefs $\mathrm{E}[\text{Profit}]$

(solid curve) against perceived market crowdedness.[7] Both Proposition 2 and Figure 4 suggest that when traders underestimate market crowdedness (to the left of the benchmark point, $\theta < \hat{\theta}$), they anticipate high trading profits, but the realized profits are lower than their expectations ($\mathrm{E}^n[\text{Profit}] > \mathrm{E}[\text{Profit}]$). Conversely, when traders overestimate market crowdedness (to the right of the benchmark point, $\theta > \hat{\theta}$), the actual expected profits are greater than expected by traders ($\mathrm{E}^n[\text{Profit}] < \mathrm{E}[\text{Profit}]$).

FIGURE 4

**Trader's ex-ante expected trading profits**

This figure presents trader's ex-ante expected trading profits under traders' beliefs ($\mathrm{E}^n[\text{Profit}]$) and true beliefs ($\mathrm{E}[\text{Profit}]$), against $\theta$ by varying $N_H$ while keeping $N$ fixed. The dashed line corresponds to the case of $\theta = \hat{\theta}$.




Intuitively, when traders underestimate market crowdedness, they underestimate the amount of private information already incorporated into prices. Consequently, they anticipate higher trading profits, only to realize that their actual profits are lower. In practice, it may take a significant amount of time for traders to recognize their errors by observing unexpectedly high or low profits. In Section IV, we will discuss other methods traders might use to learn about market

[7] In Figure 4, the default parameter values are $A = 1$, $\tau_v = 2$, $N = 40$, $\hat{N}_H = 6$ (i.e., $\hat{\theta} = 0.13$), $\rho = 0.1$, $\tau_H = 1$, and $\tau_L = 0.01$.

crowdedness and potential mistakes about its value.

Let $\hat{N} := \hat{N}_H + \hat{N}_L$ denote the actual total number of traders, out of whom $\hat{N}_H$ traders observe high precision signals and $\hat{N}_L$ traders observe low precision signals. We next study the effects on perceived and actual market depth.

The actual (objective) market clearing condition,

$$x_n + \sum_{\substack{m=1 \\ m \neq n}}^{\hat{N}} (\gamma_I \, i_m - \gamma_P \, P - \gamma_S \, S_m) = 0, \tag{18}$$

is obtained from the perceived market-clearing condition (10) by changing $N$ to $\hat{N}$. The objective residual supply schedule facing trader $n$ is

$$\hat{P}(x_n) = \frac{\gamma_I}{(\hat{N}-1)\gamma_P} \sum_{\substack{m=1 \\ m \neq n}}^{\hat{N}} i_m + \frac{\gamma_S}{(\hat{N}-1)\gamma_P} \, S_n + \frac{1}{(\hat{N}-1)\gamma_P} \, x_n. \tag{19}$$

If the actual number of traders providing liquidity differs from what is perceived, then the actual market depth will vary from the perceived market depth. Let $1/\hat{\lambda}$ and $1/\hat{\kappa}$ denote actual market depth, we have

$$\frac{1}{\hat{\lambda}} := \frac{(\hat{N}-1)\,\gamma_P}{\gamma_S}, \qquad \frac{1}{\hat{\kappa}} := (\hat{N}-1)\,\gamma_P. \tag{20}$$

The actual market depths, represented by $1/\hat{\lambda}$ and $1/\hat{\kappa}$, differ from the perceived market depths, $1/\lambda$ and $1/\kappa$ (defined in (12)) only by changing $N$ to $\hat{N}$. The values of $\gamma_S$ and $\gamma_P$ depend only on perceived parameters, including $N$. Notably, neither the subjective nor the objective market depth depends on the objective values of other parameters (e.g., $\hat{\theta}$ and $\hat{\rho}$). We have the following proposition.

**Proposition 3. Subjective and Objective Market Depth.** *The subjective and objective measure* *of market depth differ by a factor of* $(N-1)/(\hat{N}-1)$,

$$(21) \qquad \frac{1}{\lambda}=\frac{N-1}{\hat{N}-1}\,\frac{1}{\hat{\lambda}}, \qquad \frac{1}{\kappa}=\frac{N-1}{\hat{N}-1}\,\frac{1}{\hat{\kappa}}.$$

1. *When traders overestimate* $N_H$ *while keeping* $N_L$ *fixed, the perceived market depth is lower than it would be with correct beliefs, and the actual market depth is even lower (*$1/\hat{\lambda} < 1/\lambda$*);*

2. *When traders underestimate* $N_H$ *while keeping* $N_L$ *fixed, the perceived market depth is higher than it would be with correct beliefs, and the actual market depth is even higher (*$1/\hat{\lambda} > 1/\lambda$*).*

If traders overestimate the true number of market participants ($N > \hat{N}$), then while they correctly assess the demand functions of each individual trader, they overestimate the number of traders in the market and thus overall available liquidity. For example, in large markets, when traders overestimate the total number of traders by 50%, the subjective estimate of market depth is misleadingly larger than the actual market depth by about 50%. The more traders overestimate the number of market participants, the more they overestimate available liquidity.

Due to these differences, traders could theoretically learn about the actual residual supply schedule's slopes by implementing a series of experiments and analyzing price responses to trading at some off-equilibrium trading rates. In the Online Appendix B.3, we show that if traders can learn about the actual total number of traders $\hat{N}$, they can also infer the noise correlation $\hat{\rho}$ by comparing the quadratic variation of actual price dynamics with perceived price dynamics. Nevertheless, even if traders were able to estimate the number of traders ($\hat{N}$) by gathering data on residual supply schedules and subsequently deducing $\hat{\rho}$, they would still be unable to learn in real time the exact number of traders ($\hat{N}_H$) trading with them in the same direction. Consequently, our paper primarily emphasizes the scenario where the perceived fraction $\theta$ differs from the true fraction $\hat{\theta}$ of traders receiving high precision signals, while keeping $N = \hat{N}$ and $\rho = \hat{\rho}$ as fixed parameters.

# III. An Extension to Continuous-Time Model

When traders perceive markets to be crowded, they are concerned that others will be trading on similar information to their own, either before, during, or after they trade. The current accuracy of prices, which affects profitability, is a function of how much information competitors have incorporated into prices from their previous trading. To explore how perceptions of market crowdedness influence trading speed and price dynamics, we extend the static model to a dynamic model where each trader "shreds orders" and gradually adjusts his inventory in the direction of a target inventory. We then use the dynamic model to assess the implications of market crowdedness for financial stability, particularly during fire-sale events.

## A. A Continuous-Time Model

We next outline a dynamic model where all parameter values are subjective and commonly known. Each trader believes that there are $N$ strategic traders. There is a risky security with zero net supply, which pays out dividends at continuous rate $D(t)$. The dividend $D(t)$ is publicly observable and follows a stochastic process with unobservable stochastic growth rate $G^*(t)$. The dividend has a constant instantaneous volatility $\sigma_D > 0$ and constant rate of mean reversion $\alpha_D > 0$, $\mathrm{d}D(t) := -\alpha_D\, D(t)\ \mathrm{d}t + G^*(t)\ \mathrm{d}t + \sigma_D\ \mathrm{d}B_D(t)$. The growth rate $G^*(t)$ follows an AR-1 process with mean reversion $\alpha_G > 0$ and volatility $\sigma_G > 0$, $\mathrm{d}G^*(t) := -\alpha_G\, G^*(t)\ \mathrm{d}t + \sigma_G\ \mathrm{d}B_G(t)$.

Each trader $n$ observes a continuous stream of private information $I_n(t)$ about the scaled growth rate,

(22) $$\mathrm{d}I_n(t) := \tau_n^{1/2}\, \frac{G^*(t)}{\sigma_G\, \Omega^{1/2}}\, \mathrm{d}t + \rho^{1/2}\, \mathrm{d}Z(t) + (1-\rho)^{1/2}\, \mathrm{d}B_n(t).$$

Since its drift is proportional to $G^*(t)$, each increment $\mathrm{d}I_n(t)$ in the process $I_n(t)$ is a noisy observation of $G^*(t)$, scaled by $\sigma_G\, \Omega^{1/2}$ so that its variance is one. The parameter $\Omega$ measures the steady-state error variance of the trader's estimate of $G^*(t)$ in units of time; it is defined algebraically below in equation (25).

The precision parameter $\tau_n$ measures the informativeness of the signal $\mathrm{d}I_n(t)$ as a signal-to-noise ratio; it describes how fast new information flows into the market. The error terms are correlated, and $\mathrm{Cov}(\mathrm{d}I_n, \mathrm{d}I_m) = \rho\, \mathrm{d}t$ for $m \neq n$, where $\rho < 1$.

The stream of dividends contains some public information about the growth rate. Define $\mathrm{d}I_0(t) := [\alpha_D\, D(t)\, \mathrm{d}t + \mathrm{d}D(t)]/\sigma_D$ and $\mathrm{d}B_0 := \mathrm{d}B_D$. Then, public information $\mathrm{d}I_0(t)$ can be written in the form similar to the notation for private information $\mathrm{d}I_n(t)$,

(23) $$\mathrm{d}I_0(t) := \tau_0^{1/2}\, \frac{G^*(t)}{\sigma_G\, \Omega^{1/2}}\, \mathrm{d}t + \mathrm{d}B_0(t), \quad \text{where} \quad \tau_0 := \frac{\Omega\, \sigma_G^2}{\sigma_D^2}.$$

The process $I_0(t)$ is informationally equivalent to the dividend process $D(t)$. The parameter $\tau_0$ measures the precision of information revealed by the dividend process. The Brownian motions $\mathrm{d}B_0(t),\ \mathrm{d}Z(t),\ \mathrm{d}B_1(t), \ldots,\ \mathrm{d}B_N(t)$ are independently distributed.

Similar to the one-period model, traders believe that there are $N_H$ traders with private signals of high precision $\tau_H$ and $N_L$ traders with private signals of low precision $\tau_L$, with $\tau_H > \tau_L \geq 0$ and $N_L + N_H = N$. Traders are relatively overconfident in that each trader believes his own signal has high precision $\tau_H$.

The perceived fraction of other traders who observe signals of high precision (except trader $n$ himself) $\theta$ is defined as in equation (3) in the one-period model.

Let $S_n(t)$ denote the inventory of trader $n$ at time $t$. Each trader chooses a consumption intensity $c_n(t)$ and trading intensity $x_n(t)$ to maximize an expected constant-absolute-risk-aversion (CARA) utility function $U(c_n(s)) := -e^{-A\, c_n(s)}$ with risk aversion parameter $A$. Letting $\beta > 0$ denote a time preference parameter, trader $n$ solves the following maximization problem

$$\max_{[c_n(t), x_n(t)]} \mathrm{E}_t^n \left[ \int_{s=t}^{\infty} \mathrm{e}^{-\beta(s-t)}\, U(c_n(s))\, \mathrm{d}s \right], \tag{24}$$

where his inventories follow the process $\mathrm{d}S_n(t) = x_n(t)\, \mathrm{d}t$, and his money holdings $M_n(t)$ follow the process $\mathrm{d}M_n(t) = (r\, M_n(t) + S_n(t)\, D(t) - c_n(t) - P(t)\, x_n(t))\, \mathrm{d}t$. Each trader trades "smoothly" in the sense that $S_n(t)$ is a differentiable function of time with trading intensity $x_n(t) = \mathrm{d}S_n(t)/\mathrm{d}t$. Each trader explicitly takes into account how both the level of his inventory $S_n(t)$ and its derivative $x_n(t)$, or the speed of trading, affect the price $P(t)$ of a risky asset.

We next solve the model. Each trader dynamically adjusts his estimate of the growth rate and its error variance by filtering them from observables. We use $\mathrm{E}_t^n[\ldots]$ to denote the expectation of trader $n$ calculated with respect to his information at time $t$. The superscript $n$ indicates that the expectation is taken with respect to the beliefs of trader $n$. The subscript $t$ indicates that the expectation is taken with respect to trader $n$'s information set at time $t$, which consists of both private information as well as public information extracted from the history of dividends and prices. Let $G_n(t) := \mathrm{E}_t^n[G^*(t)]$ denote trader $n$'s estimate of the growth rate.

**Lemma 2.** *The "total precision" $\tau$ and the steady-state scaled error variance $\Omega$ are given by*

$$\tau := \tau_0 + \tau_H + (N-1)\frac{\left((\theta-\rho)\tau_H^{1/2}+(1-\theta)\tau_L^{1/2}\right)^2}{(1-\rho)(1+(N-1)\rho)}, \qquad \Omega := \mathrm{Var}^n\left[\frac{G^*(t)-G_n(t)}{\sigma_G}\right] = (2\,\alpha_G+\tau)^{-1}. \tag{25}$$

*Define sufficient statistics for trader n's signals and the average of other traders' signals as*

$$H_n(t) := \int_{u=-\infty}^{t} e^{-(\alpha_G+\tau)\,(t-u)}\;\mathrm{d}I_n(u), \quad n=0,1,\ldots,N, \qquad H_{-n}(t) := \frac{1}{N-1}\sum_{\substack{m=1\\ m\neq n}}^{N} H_m(t). \tag{26}$$

*Then, trader n's estimate of the growth rate $G_n(t)$ is given by*

$$\begin{aligned} G_n(t) := \mathrm{E}_t^n[G^*(t)] = \sigma_G\,\Omega^{1/2}\Big(\tau_0^{1/2}\,H_0(t) + (1-\theta)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)/(1-\rho)\;H_n(t) \\ + \frac{(\theta-\rho)\tau_H^{1/2}+(1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}\;\big(H_n(t)+(N-1)H_{-n}(t)\big)\Big). \end{aligned} \tag{27}$$

Trader $n$'s estimate $G_n(t)$ of the growth rate is conveniently written in equation (27) as the weighted sum of three sufficient statistics $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$, which summarize the information content of dividends, the trader's private information, and other traders' private information, respectively. Each trader places the same weight $\tau_0^{1/2}$ on the dividend-information signal $H_0(t)$ and assigns a larger weight to his own signal $H_n(t)$ and a lower weight to signals of $N-1$ other traders, aggregated in variable $H_{-n}(t)$. The importance of each bit of information $\mathrm{d}I_n$ about the growth rate decays exponentially at a rate $\alpha_G+\tau$, i.e., the sum of the decay rate $\alpha_G$ of fundamentals and the speed $\tau$ of learning about fundamentals, in equation (26). The steady state error variance (25) is small when the mean-reversion rate $\alpha_G$ and the amount of incoming information $\tau$ are large.

To profit from his private information, trader $n$ then exercises monopoly power in choosing how fast to demand liquidity from other traders by conjecturing the symmetric linear demand schedules for other traders $m$ ($m = 1, \ldots, N$, and $m \neq n$) given by

(28) $$x_m(t) = \frac{\mathrm{d}S_m(t)}{\mathrm{d}t} = \gamma_D\, D(t) + \gamma_H\, \breve{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t),$$

where the composite sufficient statistics $\breve{H}_m(t)$, is defined as $H_m(t) + \breve{a} H_0(t)$. The coefficient $\breve{a}$ is given in equation (A-15).

The following theorem characterizes the equilibrium trading strategies and prices. Traders calculate and dynamically update the target inventories, defined as inventory levels such that traders do not want to trade ($x_n(t) = 0$), and then trade toward these target levels smoothly to optimize the market impact costs of their trading.

**Theorem 2. Equilibrium of the Continuous-time Model with Imperfect Competition.** *There exists a steady-state equilibrium with symmetric linear flow-strategies and positive trading volume if and only if the six polynomial equations (A-31)–(A-36) have a solution satisfying the second-order condition $\gamma_P > 0$ and the stationarity condition $\gamma_S > 0$. Such an equilibrium has the following properties:*

1. *There is an endogenously determined constant $C_L > 0$, defined in equation (A-29), such that $S_n^{TI}(t)$ is trader $n$'s "target inventory" defined as $S_n^{TI}(t) = C_L\,(H_n(t) - H_{-n}(t))$, and trader $n$'s optimal flow-strategy $x_n(t)$ is given by*

   (29) $$x_n(t) = \frac{\mathrm{d}S_n(t)}{\mathrm{d}t} = \gamma_S\,\left(S_n^{TI}(t) - S_n(t)\right),$$

   *the equilibrium trading strategies and target inventory levels depend solely on traders' subjective beliefs rather than objective beliefs.*

2. *Traders believe that the equilibrium price is*

(30)
$$P(t) = \frac{D(t)}{r + \alpha_D} + C_G \frac{\sum_{n=1}^{N} G_n(t)}{N(r + \alpha_D)(r + \alpha_G)},$$

*where the endogenous parameter* $C_G > 0$*, defined in equation (A-29), and* $\frac{1}{N}\sum_{n=1}^{N} G_n(t)$ *is traders' average expected growth rate.*

Extensive numerical calculations suggest that the existence condition for the continuous-time model is exactly the same as the existence condition for the one-period model, presented in equation (13). Trader $n$ trades smoothly from current inventories toward target inventories, and the parameter $\gamma_S$ sets the rate of convergence. Target inventories are dynamically changing; they are positive if trader $n$'s signal is greater than the average signal of other traders, and they are negative otherwise. The price immediately reveals the average of all private signals by revealing the average estimate of the dividend growth rate.

If parameter $C_G$ were equal to one, then the price would be equal to the average of traders' risk-neutral buy-and-hold valuations, consistent with the Gordon's growth formula. The aggregation of heterogeneous beliefs in a dynamic setting makes the multiplier $C_G$ to be less than one. The dampening effect $C_G < 1$ occurs because traders internalize their future disagreement about the growth rate. Traders engage in short-term speculative trading which dampens prices relative to the fundamental values. The equilibrium of the similar continuous-time model, but with perfect competition, is presented in the Online Appendix B.4.

## B. Implications of Dynamic Model

The correlation between the sufficient statistics for trader $n$'s signals and the average signals of others, based on traders' beliefs, is denoted by $\text{Corr}^n[H_n(t), H_{-n}(t)]$. Intuitively, when traders' private signals about the fundamental value of the asset become highly correlated, it is more likely that traders will employ similar trading strategies, making the market more crowded. We study the effects of market crowdedness by varying the fraction of traders with high-precision signals ($\theta$).

Similar to the analytical results of the one-period model, when traders perceive a high fraction of traders with high-precision signals, they trade less aggressively and aim for smaller target positions. Conversely, a lower perceived fraction of traders with high-precision signals leads to more aggressive trading. These results are presented in Appendix B.2.

In the dynamic model, on the one hand, traders want to trade slowly to reduce the temporary price impact costs from adverse selection. On the other hand, they want to trade quickly before the permanent price impact of competitors trading on similar information makes profit opportunities go away. So each trader "shreds orders" and trades smoothly from current inventories toward target inventories. The degree of market crowdedness affects the speed at which traders' inventories converge to target levels and their size.

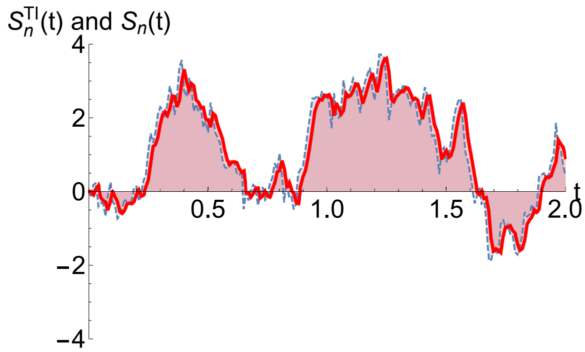
Figure 5 plots two simulated paths for target inventories (dashed curve) and actual inventories (solid curve) from the perspective of each trader.[8] In panel (a), traders perceive a less crowded market. Consequently, both perceived temporary and permanent price impacts are relatively low. As a result, traders trade swiftly, set larger target inventories (as also illustrated in

[8]The paths are generated using equation (A-21), which describe the dynamics of composite sufficient statistics $\check{H}_n(t), \check{H}_{-n}(t)$, and $S_n^{TI}(t)$. Numerical calculations in Figure 5 are based on the exogenous parameter values $r$, $A$, $\alpha_D$, $\alpha_G$, $\sigma_D$, $\sigma_G$, $\rho$, $\tau_L$ presented in Table 1 in Section A. Parameter values $\tau_H = 1$, $\rho$, and $N = 60$ in both panels (a) and (b); $N_H = 3$ and $N_L = 57$ in panel (a); $N_H = 20$ and $N_L = 40$ in panel (b).

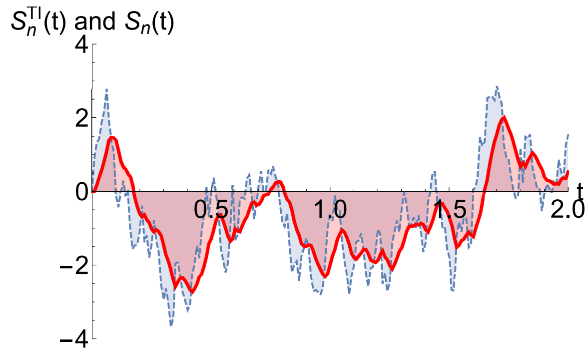
Figure B-4), exhibit higher trading aggressiveness, and their actual inventories remain close to target levels. In panel (b), traders perceive a more crowded market. This perception leads to both the temporary and permanent price impacts being relatively high. As a consequence, traders trade more slowly, set smaller target inventories, show reduced trading aggressiveness, and their actual inventories deviate significantly from target levels.

FIGURE 5

**Simulated paths for target inventories $S_n^{TI}(t)$ and actual inventories $S_n(t)$**




(a) Less crowded markets




(b) More crowded markets

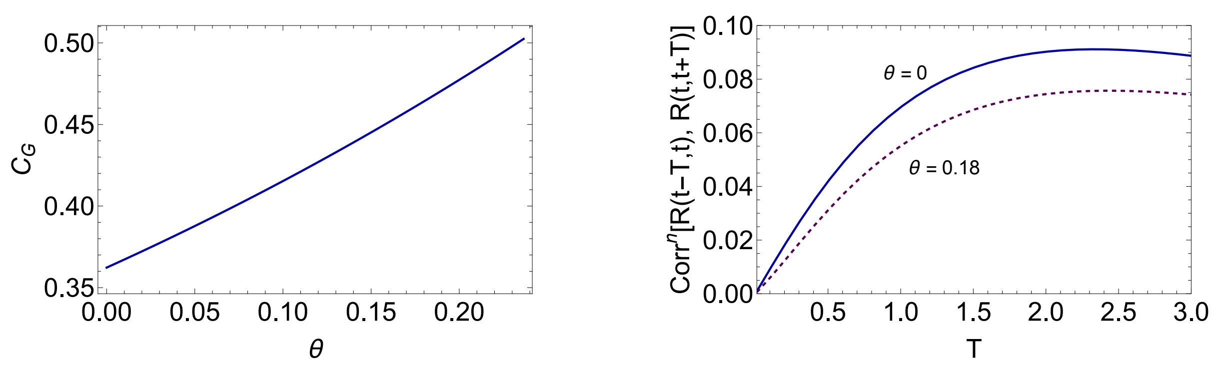
In the model, returns exhibit time-series momentum, and the degree of perceived market crowdedness affects its magnitude. The dampening effect $C_G < 1$ occurs because traders internalize their future disagreement about the growth rate. Each trader agrees to disagree with others about how to interpret private signals at the present and how to interpret private signals in the future. When internalized, this disagreement about future dampens price fluctuations at present and injects predictability going forward. Each trader thinks that other traders make erroneous interpretation of their private and public information about fundamentals. He tries to profit on this belief by trading ahead of anticipated short-term revisions of others' expectations. The short-term speculative trading dampens prices relative to the average of traders' long-term valuations. Price dampening effect leads to time-series momentum.

When perceived market crowdedness is low, traders tend to trade more aggressively on short-run profit opportunities making price dampening more significant (i.e., $C_G$ is smaller) and return momentum more pronounced.

Figure 6 depicts price dampening parameter and the autocorrelations of the cumulative returns as perceived by traders, $\text{Corr}^n\left[R(t-T,t), R(t,t+T)\right]$, for different period $T$.[9] The derivations are presented in the Online Appendix Section B.5. All return autocorrelations are positive, implying that the returns exhibit time-series return momentum. The left panel shows that price dampening is larger (i.e., $C_G$ is smaller) when the perceived market crowdedness is low. The right panel shows that time-series momentum, as measured by autocorrelations of cumulative returns, is more pronounced when the perceived market crowdedness is low.[10]

FIGURE 6

**Parameter value $C_G$ and autocorrelations of returns, $\text{Corr}^n[R(t-T,t), R(t,t+T)]$**




This result is broadly in line with Kyle et al. (2023) who analyze a discrete-time dynamic

[9]In Figure 6, parameter values $\beta = 0.05$, $\tau_H = 1$, and $N_H = 4$. Parameter values $r$, $A$, $N$, $\alpha_D$, $\alpha_G$, $\sigma_D$, $\sigma_G$, $\rho$, $\tau_L$ are as presented in Table 1 in Section A.

[10]It can be shown that return can also exhibit positive autocorrelation under objective beliefs. Kyle, Obizhaeva, and Wang (2023) provide a detailed analysis of return autocorrelation under objective beliefs.

trading model with competitive traders who agree to disagree about the precision of private information and show that the price dampening is stronger and thus time-series return momentum is more pronounced when trading opportunities are more frequent. This is also in line with Lee and Swaminathan (2000) and Cremers and Pareek (2014) who find empirically that time-series momentum patterns are stronger for stocks with higher volume and more short-term trading.

We place a "hat" over a parameter to denote its objective value when this value is potentially different from the subjective value. Let $\hat{\theta}$ denote the true (objective) fraction of traders receiving signals of high precision, which may differ from the perceived fraction $\theta$.

The expected profit from trading is measured by the unconditional expected profit per unit time, denoted as Profit $:= (\mathrm{d}P(t) + D(t)\mathrm{d}t - rP(t)\mathrm{d}t)S_n(t)$, where the return dynamics $\mathrm{d}P(t) + D(t)\mathrm{d}t - rP(t)\mathrm{d}t$ is presented in equation (B-27) in the Online Appendix. The inventory of traders, $S_n(t)$, follows the process $\mathrm{d}S_n(t) = x_n(t)\ \mathrm{d}t$, where $x_n$ represents the optimal flow trading, as presented in Theorem 2. We compute the expected profit of trading under traders' beliefs ($\mathrm{E}^n[\text{Profit}]$), and under objective beliefs, $\mathrm{E}[\text{Profit}]$.

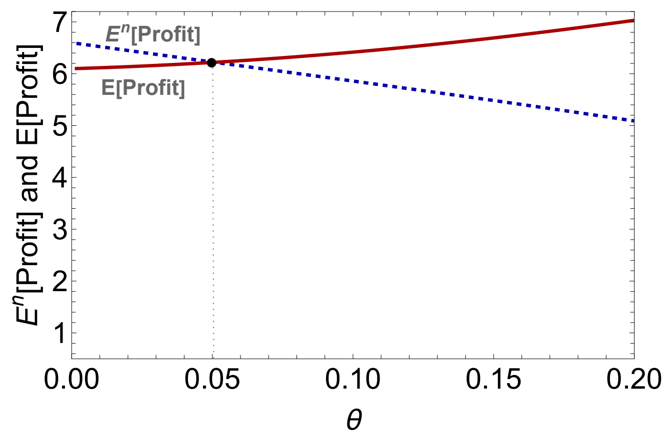
Figure 7 depicts the expected profit of trading under traders' beliefs (dashed curve) and the expected profit of trading under true beliefs (solid curve) against perceived market crowdedness.[11] When traders underestimate market crowdedness, they underestimate the information already impounded into prices, and as a result, the actual expected profit of trading is lower than that expected by traders ($\mathrm{E}^n[\text{Profit}] > \mathrm{E}[\text{Profit}]$). When traders overestimate market crowdedness, the actual expected profit of trading may be greater than expected by traders ($\mathrm{E}^n[\text{Profit}] < \mathrm{E}[\text{Profit}]$). Thus, these results about profits continue to hold in dynamic settings.

[11] Parameter values $r$, $N$, $A$, $\alpha_D$, $\alpha_G$, $\sigma_D$, $\sigma_G$, $\rho$, and $\tau_L$ are presented in Table 1 in Section A, $\tau_H = 1$, and $\hat{N}_H = 4$.

FIGURE 7

**Expected profit of trading**

This figure presents the expected profit of trading under subjective (dashed curve) and objective (solid curve) beliefs against $\theta$ by varying $N_H$ while keeping $N$ fixed. The dashed line corresponds to the case of $\theta = \hat{\theta}$.




# IV. Market Fragility and Inferring Crowdedness

It can be difficult for traders to infer their mistakes by observing market price dynamics.[12] In this section, we discuss how the degree of market crowdedness and traders' perceptions of it can be inferred by examining price dynamics during fire-sale events. We also examine how market crowdedness influences market fragility, specifically the market's vulnerability to rapid price changes.[13]

## A. Inferring Market Crowdedness

Assume traders may not accurately estimate the proportion of their peers receiving high-precision signals, i.e., $\theta \neq \hat{\theta}$, while keeping $N$ constant. We then explore what insights can be learned about

[12]See Section B.3 of the Online Appendix for details.

[13]See Goldstein, Huang, and Yang (forthcoming) for a review of the theoretical literature on financial market fragility.

market crowdedness and traders' mistakes by examining price dynamics during fire-sale events. As an illustrative example, we study the Quant meltdown of 2007 under the assumption that it occurred due to the liquidation of a large position in a "fire sale" mode. This approach can similarly be applied to other fire-sale episodes.

The first half of August 2007 was an exceptionally volatile period for many market-neutral quantitative equity strategies. In the preceding months, these strategies had gained popularity among asset managers and attracted large inflows of funds. In early August, two multi-strategy Bear Stearns hedge funds faced substantial losses due to their investments in subprime mortgages. These hedge funds urgently needed to raise liquidity and presumably liquidated parts of their portfolios invested in relatively liquid market-neutral quantitative equity strategies.

These fire sales led to a substantial increase in correlation across various conventional market-neutral quantitative equity strategies, followed by a large drop in prices and a subsequent V-shaped recovery.

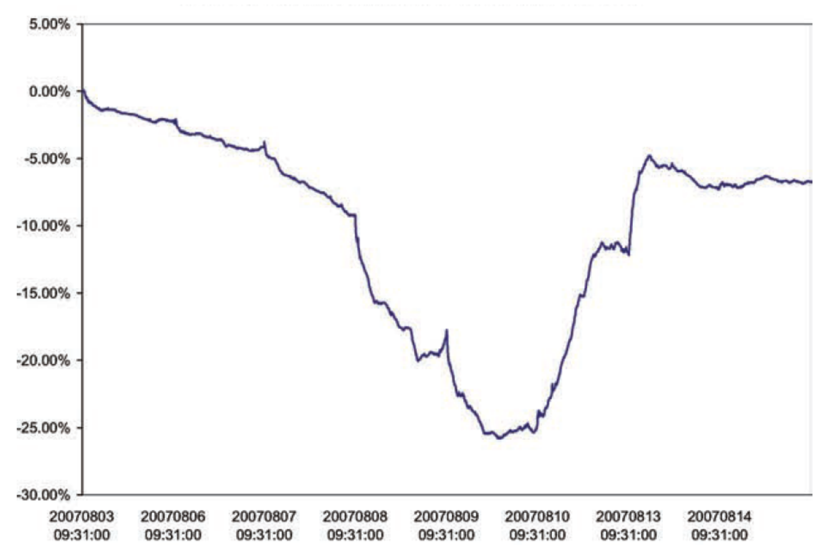
Figure 8 shows the cumulative return to a long-short market-neutral value and momentum strategy for U.S. large-cap stocks during August 3 through 14, 2007. The cumulative returns dropped by about 27 percent from August 3 to August 9, but recovered to almost initial levels in about 3 business days by August 14.

We calibrate the parameters of our model to match these two numbers: 27 percent and 3 days. Following the interpretation of Stein (2009), we model this event as a potential outcome when some traders may misestimate the proportion of their peers following the same strategies.

For simplicity, suppose that at the starting point of time 0 and afterwards, the signals of all traders except trader $n$, as well as dividends, are at their long-term mean of zero; that is, signals $H_{-n}(t) = 0$ and dividends $D(t) = 0$ with $H_0(t) = 0$ for $t \geq 0$. Trader $n$ observes a private signal

FIGURE 8

**A Minute-by-Minute Data during the Quant Event in 2007 from Pedersen (2009)**




$H_n(0) > 0$ at time 0 and holds a positive inventory $S_n(0)$ equal to the target level $S_n^{TI}(0)$, as given by Theorem 2:

$$S_n(0) = S_n^{TI}(0) = C_L\, H_n(0) > 0. \tag{31}$$

Equation (A-19) yields the price at time $t = 0$,

$$P(0) = \frac{\gamma_S(\theta)}{(N-1)\gamma_P(\theta)}\, S_n(0) > 0. \tag{32}$$

At time $t = 0^+$, trader $n$ receives new private information $H_n(0^+) = 0$, and his target inventory changes from $S_n^{TI}(0) > 0$ to $S_n^{TI}(0^+) = 0$.[14] We examine the price reaction if trader $n$ begins to trade toward his new target inventory at the speed of $\bar{x}_n(t)$ starting at $t = 0^+$, based on

[14]We model the shock to target inventories as a response to the arrival of new information, but it could also be interpreted as an exogenous liquidity shock. The analysis is similar to that in Section 4.2 of Kyle et al. (2018). Our primary objective is to identify potential misestimates of market crowdedness, a topic not studied by Kyle et al. (2018).

his possibly incorrect beliefs about market crowdedness ($\bar{\theta}$):

$$\bar{x}_n(t) = \bar{\gamma}_S(\bar{\theta}) \left( S_n^{TI}(t) - \bar{S}_n(t) \right). \tag{33}$$

The fire-sale trading intensity $\bar{\gamma}_S(\bar{\theta})$ is based on traders' incorrect beliefs $\bar{\theta}$ about market crowdedness $\hat{\theta}$, with $\bar{\theta} < \hat{\theta}$. The fire-sale trading intensity is greater than the optimal equilibrium trading intensity $\gamma_S(\hat{\theta}) := \gamma_S(\theta)|_{\theta=\hat{\theta}}$. Since $\mathrm{d}\bar{S}_n(t) = \bar{x}_n(t)\mathrm{d}t$, and using equation (33), it follows that the trader's inventory $\bar{S}_n(t)$ evolves along the trajectory:

$$\bar{S}_n(t) = \mathrm{e}^{-\bar{\gamma}_S(\bar{\theta})\, t} \left( S_n(0) + \int_{u=0}^{t} \mathrm{e}^{\bar{\gamma}_S(\bar{\theta})\, u}\, \bar{\gamma}_S(\bar{\theta})\, C_L \left( H_n(u) - H_{-n}(u) \right)\, \mathrm{d}u \right). \tag{34}$$

Equation (34) implies that trader $n$'s expected inventory $\mathrm{E}_0^n[\bar{S}_n(t)]$ as

$$\mathrm{E}_0^n[\bar{S}_n(t)] = \mathrm{e}^{-\bar{\gamma}_S(\bar{\theta})\, t}\, S_n(0). \tag{35}$$

This rushed sale, based on a possibly incorrect estimate of market crowdedness, leads to a sharp drop in price. Using equations (A-19), (33), and (35), trader $n$ can calculate the impulse-response functions of how market prices $\mathrm{E}_0^n[\bar{P}(t)]$ are expected to change in response:

$$\begin{aligned} \mathrm{E}_0^n[\bar{P}(t)] &= \frac{\gamma_S(\theta)}{(N-1)\gamma_P(\theta)}\, \mathrm{E}_0^n[\bar{S}_n(t)] + \frac{1}{(N-1)\gamma_P(\theta)}\, \mathrm{E}_0^n[\bar{x}_n(t)] \\ &= -\frac{\bar{\gamma}_S(\bar{\theta}) - \gamma_S(\theta)}{(N-1)\gamma_P(\theta)}\, \mathrm{e}^{-\bar{\gamma}_S(\bar{\theta}) t}\, S_n(0). \end{aligned} \tag{36}$$

If selling occurs at the optimal speed ($\gamma_S(\hat{\theta}) := \gamma_S(\theta)|_{\theta=\hat{\theta}}$) based on true beliefs about market crowdedness $\hat{\theta}$, then the price will drop to its equilibrium level of 0 instantaneously at $t = 0^+$ and remain at that level. If selling is speeded up, then price path will exhibit V-shaped patterns. The fire-sale trading intensity is greater than the optimal equilibrium trading intensity

TABLE 1

**The given and calibrated parameter values**

| Parameter | Description | Value |
|---|---|---|
| $r$ | Risk-free rate | 0.01 |
| $A$ | Risk aversion | 1 |
| $N$ | Numbers of traders | 60 |
| $\rho$ | Correlation of noise | 0.1 |
| $\alpha_D$ | Mean-reversion rate of dividend | 0.06 |
| $\sigma_D$ | Instantaneous volatility of dividend | 0.5 |
| $\alpha_G$ | Traders' mean-reversion rate of $G(t)$ | 0.5 |
| $\sigma_G$ | Instantaneous volatility of $G(t)$ | 0.2 |
| $\tau_H$ | Precision of trader $n$'s signal | 3.6 |
| $\tau_L$ | Precision of others' signal | 0.01 |
| $\hat{\theta}$ | Actual fraction of traders with high precision signal | 8.5% |
| $\gamma_S(\hat{\theta})$ | Optimal trading rate based on $\hat{\theta}$ | 90.96 |
| $\bar{\theta}$ | Underestimated fraction of traders with high precision signal | 5.4% |
| $\bar{\gamma}_S(\bar{\theta})$ | Fire-sale trading rate based on $\bar{\theta}$ | 115.52 |

$\gamma_S(\hat{\theta})$. Equations (36) and (32) then imply that

$$\frac{\mathrm{E}_0^n[\bar{P}(t)]}{P(0)} = \left(1 - \frac{\bar{\gamma}_S(\bar{\theta})}{\gamma_S(\hat{\theta})}\right) \mathrm{e}^{-\bar{\gamma}_S(\bar{\theta})t}, \quad t > 0. \tag{37}$$

Next, we use equation (37) to calibrate the model parameters by matching the predicted price path to the realized price path during the quant meltdown event. Specifically, we match the empirical price drop of 27% and the recovery time of three business days, as depicted in Figure 8. The half-life of recovery of 1.5 business days in equation (37) implies $t = 1.5/251 = 0.006$, given that there are 251 business days in 2007. Consequently, the fire-sale rate is $\bar{\gamma}_S(\bar{\theta}) = \ln 2/0.006 = 115.52$. The price drop of 27% in equation (37) implies that the execution speed is 1.27 times faster than optimal, *i.e.*, $\bar{\gamma}_S(\bar{\theta})/\gamma_S(\hat{\theta}) = 1.27$. Thus, we calibrate the model parameters to match the fire-sale rate $\bar{\gamma}_S(\bar{\theta}) = 115.52$ and the optimal execution rate $\gamma_S(\hat{\theta}) = 115.52/1.27 = 90.96$.

We first fix a set of parameters. We assume $A = 1$ since the risk aversion parameter $A$ scales trading volume but has no effect on prices.[15] We assume $r = 0.01$ and $\alpha_D = 0.06$, implying $1/(r + \alpha_D) = 14.3$. We also assume the dividend volatility $\sigma_D = 0.5$ and the instantaneous volatility of the growth rate $\sigma_G = 0.2$.[16] The mean-reversion rate $\alpha_G = 0.5$ implies that the growth rate $G(t)$ mean reverts in about $\ln(2)/0.5 \approx 1.4$ years. We assume $N = 60$, $\rho = 0.1$, $\tau_H = 3.6$, $\tau_L = 0.01$, and $\alpha_G = 0.5$; traders are overconfident, and they actively engage in short-term trading. All exogenous parameters are presented at the top part of Table 1.

We then use theoretical predictions of equation (37) to calibrate the remaining parameters — traders' incorrect beliefs about the fraction of traders with high precision signal $\bar{\theta}$ and its true value $\hat{\theta}$—by matching empirical magnitude of price drop of 27% and recovery time of three business days. As discussed in the previous paragraph, the half-life of recovery (1.5 business days) in equation (37) implies a fire-sale rate of $\bar{\gamma}_S(\bar{\theta}) = \ln 2/0.006 = 115.52$. The execution speed of $\bar{\gamma}_S(\bar{\theta})$ corresponds to traders' belief of $\bar{\theta} = 5.4\%$. The price drop of 27% in equation (37) then implies an optimal trading rate of $\gamma_S(\hat{\theta}) = 90.96$, which corresponds to a market crowdedness of $\hat{\theta} = 0.085$.

Thus, this exercise suggests that the observed price patterns could be consistent with the market where 8.5% of traders observe high-precision signals ($\hat{\theta} = 8.5\%$), while traders incorrectly believe that only 5.4% of traders observe high-precision signals ($\bar{\theta} = 5.4\%$). As a result, they execute a large order, speeding up execution by about 27% relative to the optimal rate. Our illustrative example demonstrates that market crowdedness is a critical factor, as even a small

[15] We show in Appendix A.7 that if risk aversion $A$ is scaled by a factor of $F$ to $A/F$, then $C_L$ changes to $C_L\,F$, $\lambda$ changes to $\lambda/F$, $\kappa$ changes to $\kappa/F$, $S_n^{TI}(t)$ changes to $S_n^{TI}(t)\,F$, while $\gamma_S$, $C_G$, and the price remain unchanged.

[16] If the growth rate is zero, then $P(t) = D(t)/(r + \alpha_D)$. We assume that a firm pays out all earnings as dividends, then the P/E ratio is approximately $1/(r + \alpha_D)$. The median of S&P 500 P/E ratio is 14.85. Rountree, Weston, and Allayannis (2008) document that the average (median) standard deviation of quarterly earnings per share is 0.72 (0.19).

misestimation can result in significant price disruption.

## B. Market Crowdedness and Fragility

Many traders and regulators are concerned about fragility of crowded markets. With model calibrated in Section A, we could study how flash-crash price patterns would depend on market crowdedness.

Panel (a) of Figure 9 matches the empirical pattern illustrated in Figure 8 for the case where the fraction of traders with high-precision signals is $\hat{\theta} = 0.085$. If selling is at the optimal speed, $\gamma_S(\hat{\theta}) = \gamma_S^a = 90.96$, then price drops instantaneously to its equilibrium level of 0 and remains at that level (dashed line). When selling is rushed relative to optimal speed and $\bar{\gamma}_S(\bar{\theta}) = 115.52 > \gamma_S^a = 90.96$, the price drops by about 27% below its equilibrium level, and it takes about three business days for it to recover, as happened during the quant meltdown (solid line).

Suppose next that the market is more crowded, and $\hat{\theta} = 0.150$ instead of $\hat{\theta} = 0.085$. Then, the optimal selling rate $\gamma_S(\hat{\theta}) = \gamma_S^b = 61.70$ instead of $\gamma_S(\hat{\theta}) = 90.96$. Panel (b) of Figure 9 depicts what would be the price dynamics in this case. Suppose a trader executes his order with the same fire-sale trading intensity $\bar{\gamma}_S(\bar{\theta}) = 115.52$ as in panel (a), then the half-life of price recovery remains the same, but the price drop will be equal to 87 percent, i.e., several times more pronounced than 27% (solid line). When market is more crowded, traders are more cautious about providing liquidity to others and price drops are more pronounced.

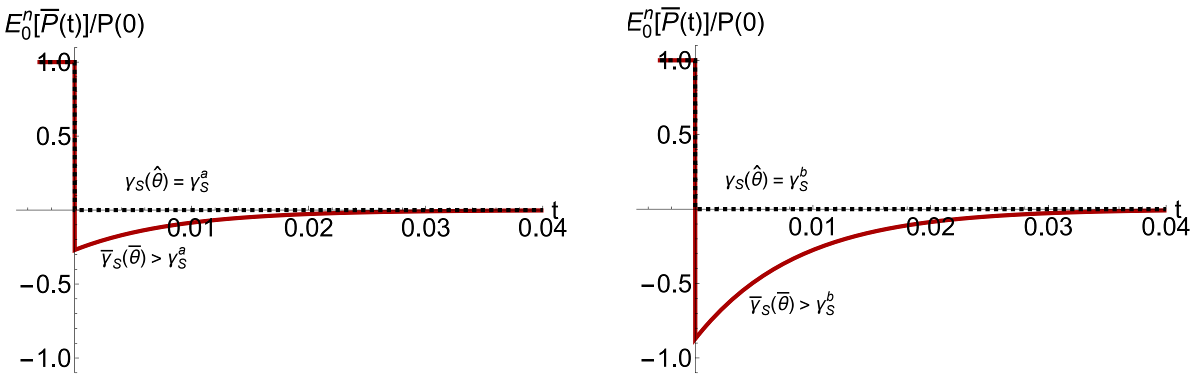
Alternatively, one may speed up the execution of orders by the same factor relative to the optimal rate, for example, $\bar{\gamma}_S(\bar{\theta}) = 115.52$ instead of $\gamma_S(\hat{\theta}) = \gamma_S^a = 90.96$ in panel (a) and $\bar{\gamma}_S(\bar{\theta}) = 78.36$ instead of $\gamma_S(\hat{\theta}) = \gamma_S^b = 61.70$ in panel (b).[17] Then, equation (37) implies that the

[17] In both cases, the execution speed is $\bar{\gamma}_S(\bar{\theta})/\gamma_S(\hat{\theta}) = 115.52/90.96 = 78.36/61.70 = 1.27$ times faster than normal

FIGURE 9

**Price dynamics of expected prices $\mathrm{E}_0^n[\bar{P}(t)]$**

This figure presents the price dynamics of expected prices $\mathrm{E}_0^n[\bar{P}(t)]$ under optimal execution (dashed line) and rushed execution (solid line) for markets with different levels of market crowdedness. Execution speed $\bar{\gamma}_S(\bar{\theta}) = 115.52$ is the same in both panels.




(a) Calibrated Price Dynamics (b) Price Dynamics in a More Crowded Market

initial price drop will be the same in both markets, but the rate of price recovery would be much slower in more crowded markets in panel (b), because traders are more cautious (not depicted).

In summary, price dislocations are expected to be more pronounced and price recoveries slower in more crowded markets. Consistent with our model's predictions, Brown et al. (2022) find that, during periods of industry distress, crowded stocks experience larger price declines followed by slower recoveries.

# V. Conclusion

Since the Quant Meltdown of August 2007, institutional traders have been increasingly concerned about crowded markets, because this may impede their efforts to deliver good performance and make them vulnerable to externalities imposed by other market participants.

speed.

We develop a model of trading among strategic informed traders to study the phenomenon of crowded markets, where traders may hold incorrect views about the proportion of those observing high-precision signals. These misperceptions influence their beliefs about the correlation of private signals related to the fundamentals observed by traders pursuing the same strategy.

If traders underestimate the market crowdedness, traders trade intensively, take large positions, and don't shirk from providing liquidity to others. They underestimate the amount of information impounded into prices, and as a result, their actual profits are lower than expected.

Our results suggest that elevated levels of market crowdedness are likely to be associated with a combination of high levels of institutional ownership (e.g., shares held by hedge funds), low rates of share turnover, and low market liquidity.

Traders' misestimation of market crowdedness distorts their optimal trading strategies and market prices. When some traders liquidate large positions at a fire sale pace based on their beliefs about market crowdedness, flash crashes can occur. We show that flash-crash price patterns are expected to be more pronounced and price recoveries are slower when markets are more crowded.

The magnitude of price drops and the speed of recovery during fire-sale events can help us identify the degree of market crowdedness and traders' perceptions of this parameter. It may be important for regulators to monitor the risks of major investment strategies getting overcrowded as well as pay attention on whether market participants become concerned about this possibility.

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# A. Proofs

## A.1. Proof of Lemma 1

Each trader $n$ observes signal $i_n$ and $i_{-n}$ defined in equations (1) and (4),

$$\mathrm{E}^n[v] = \mathrm{E}\left[ v \middle| v + \frac{\rho^{1/2}z+(1-\rho)^{1/2}e_n}{\tau_H^{1/2}\,\tau_v^{1/2}},\ v + \frac{\rho^{1/2}z+(1-\rho)^{1/2}/(N-1)\sum_{m=1,m\neq n}^{N} e_m}{(\theta\tau_H^{1/2}+(1-\theta)\tau_L^{1/2})\,\tau_v^{1/2}} \right],$$

$$\mathrm{Var}^n[v] = \mathrm{Var}\left[ v \middle| v + \frac{\rho^{1/2}z+(1-\rho)^{1/2}e_n}{\tau_H^{1/2}\,\tau_v^{1/2}},\ v + \frac{\rho^{1/2}z+(1-\rho)^{1/2}/(N-1)\sum_{m=1,m\neq n}^{N} e_m}{(\theta\tau_H^{1/2}+(1-\theta)\tau_L^{1/2})\,\tau_v^{1/2}} \right].$$

Equations (6) and (7) in Lemma 1 follow directly from the projection theorem for jointly normally distributed random variables.

## A.2. Proof of Theorem 1

Conjecturing that other traders submit symmetric linear demand schedules,

$$X_m(i_m, S_m, P) = \gamma_I\, i_m - \gamma_P\, P - \gamma_S\, S_m, \qquad m = 1, \ldots, N, \quad m \neq n, \tag{A-1}$$

and believing the market-clearing condition holds in equilibrium,

$$x_n + \sum_{\substack{m=1 \\ m \neq n}}^{N} (\gamma_I\, i_m - \gamma_P\, P - \gamma_S\, S_m) = 0, \tag{A-2}$$

each trader $n$ infers a perceived residual supply schedule $P(x_n)$ that he faces as a function of quantity $x_n$ he trades:

$$P(x_n) = \frac{\gamma_I}{\gamma_P}\, i_{-n} + \frac{\gamma_S}{(N-1)\gamma_P}\, S_n + \frac{1}{(N-1)\gamma_P}\, x_n. \tag{A-3}$$

Substituting $P(x_n)$ from equation (A-3) into the utility maximization (5), which is equivalent to maximizing

$$\mathrm{E}^n[W_n] - \tfrac{1}{2} A\, \mathrm{Var}^n[W_n], \tag{A-4}$$

where the mean and variance are given by

$$\mathrm{E}^n[W_n] = \mathrm{E}^n[v]\, (S_n + x_n) - P(x_n)\, x_n, \qquad \mathrm{Var}^n[W_n] = (S_n + x_n)^2\, \mathrm{Var}^n[v]. \tag{A-5}$$

Taking the first-order derivative with respect to $x_n$ in (A-4) yields the optimal demand $x_n$:

$$x_n = \frac{\mathrm{E}^n[v] - \frac{\gamma_I}{\gamma_P}\, i_{-n} - \left(\frac{\gamma_S}{(N-1)\gamma_P} + \frac{A}{\tau}\right) S_n}{\frac{2}{(N-1)\gamma_P} + \frac{A}{\tau}}. \tag{A-6}$$

From equation (A-3), we have

(A-7) $$i_{-n} = \frac{\gamma_P}{\gamma_I}\left(P - \frac{\gamma_S}{(N-1)\gamma_P}\, S_n + \frac{1}{(N-1)\gamma_P}\, x_n\right).$$

Substituting $\mathrm{E}^n[v]$ from equation (7) and $i_{-n}$ from (A-7), into equation (A-6) above, and then solving for $x_n$ as a function of price $P$, $i_n$, and $S_n$. In a symmetric linear equilibrium, the strategy chosen by trader $n$ must be the same as the linear strategy (9) conjectured for the other traders. Equating the corresponding coefficients of the variables $i_n$, $P$, and $S_n$ yields a system of four equations in terms of the three unknowns $\gamma_I$, $\gamma_P$, and $\gamma_S$. The unique solution is

(A-8) $$0 < \gamma_S = \frac{2+(N-2)\rho}{1+(N-1)\rho} - \frac{N(1-\rho)\tau_H^{1/2}}{(N-1)(1-\theta)(1+(N-1)\rho)(\tau_H^{1/2}-\tau_L^{1/2})} < 1,$$

(A-9) $$\gamma_P = \frac{\tau(1+(N-1)\rho)}{(1+(N-1)\theta)\tau_H^{1/2}+(N-1)(1-\theta)\tau_L^{1/2}}\, \frac{\gamma_I}{\tau_v^{1/2}}, \qquad \gamma_I = \frac{(1-\theta)(\tau_H^{1/2}-\tau_L^{1/2})}{A(1-\rho)}\tau_v^{1/2}\, \gamma_S.$$

Substituting (A-8) and (A-9) into (A-6) yields trader $n$'s optimal demand (15). Substituting (15) into (11) yields the equilibrium price (16). The second-order condition has the correct sign if and only if $\frac{2}{(N-1)\gamma_P} + \frac{A}{\tau} > 0$. This is equivalent to

$$\theta < 1 - \frac{N(1-\rho)\tau_H^{1/2}}{(N-1)(2+(N-2)\rho)(\tau_H^{1/2}-\tau_L^{1/2})} < 1.$$

This concludes the proof.

## A.3. Proof of Proposition 1

From equation (A-8), we get the partial derivative of $\gamma_S$ with respect to $\theta$:

(A-10) $$\frac{\mathrm{d}\gamma_S}{\mathrm{d}\theta} = -\frac{N(1-\rho)\tau_H^{1/2}}{(N-1)(1-\theta)^2(1+(N-1)\rho)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)} < 0.$$

Similarly, it is straightforward to show

$$\frac{\mathrm{d}\gamma_P}{\mathrm{d}\theta} < 0, \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}\theta} > 0, \qquad \frac{\mathrm{d}\kappa}{\mathrm{d}\theta} > 0, \qquad \text{and} \qquad \frac{\mathrm{d}\,\mathrm{E}^n[|S_n^{TI}|]}{\mathrm{d}\theta} < 0.$$

Assume the initial inventory $S_n = 0$. Trader $n$'s expected trading profit is

$$\mathrm{E}^n[\text{Profit}] := \mathrm{E}^n\Big[(\mathrm{E}^n[v] - P)x_n\Big]$$
$$= \frac{\tau_v \gamma_S (\tau_H^{1/2} - \tau_L^{1/2})^2 (1-\theta)^2 (N-1)^2}{A\, N^2\, (1-\rho)^2\, \tau} \Big(\Big(\tau_H^{1/2} - \tau_L^{1/2}\Big)^2 (1-\theta)^2 + (1-\rho)\Big(1 + \tfrac{1}{N-1}\Big)\Big),$$

(A-11)

where $\mathrm{E}^n[v] - P = \frac{\tau_v^{1/2}(\tau_H^{1/2}-\tau_L^{1/2})(N-1)(1-\theta)}{\tau N(1-\rho)}(i_n - i_{-n}), \quad x_n = \frac{\gamma_S \tau_v^{1/2}(\tau_H^{1/2}-\tau_L^{1/2})(1-\theta)(N-1)}{A(1-\rho)N}(i_n - i_{-n}).$

Under traders' beliefs, $\mathrm{E}^n[(i_n - i_{-n})^2] = \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 (1-\theta)^2 + (1-\rho)\left(1 + \frac{1}{N-1}\right)$. Since $\frac{\mathrm{d}\gamma_S}{\mathrm{d}\theta} < 0$ and direct calculation shows that $\frac{1}{\tau}\Big(\Big(\tau_H^{1/2} - \tau_L^{1/2}\Big)^2 (1-\theta)^2 + (1-\rho)\Big(1 + \frac{1}{N-1}\Big)\Big)$ decreases with $\theta$, it follows that $\mathrm{E}^n[\text{Profit}]$ decreases with $\theta$.

To show that $\gamma_S$ equals to the ratio of the trading quantity in the equilibrium with imperfect competition to that with perfect competition, we now solve the competitive equilibrium. Assume market is perfectly competitive and $\tau_H > \tau_L$. The first-order condition of utility maximization (5) yields the optimal demand, $x_n = (\mathrm{E}^n[v] - P)\frac{\tau}{A} - S_n$. The market-clearing condition (A-2) yields that the equilibrium price, $P^c = \frac{1}{N}\sum_{n=1}^{N} \mathrm{E}^n[v]$.

Therefore, there exists a unique symmetric equilibrium with linear trading strategies and nonzero trade. Trader $n$ chooses the quantity $x_n^c = S_n^{TI} - S_n$ (equation (15) with $\gamma_S = 1$). The price $P^c$ is the same as with imperfect competition $P$ (equation (16)). This implies that $0 < \gamma_S = x_n / x_n^c < 1$ equals to the ratio of the trading quantity in the equilibrium with imperfect competition to that with perfect competition. This ratio captures the level of market competitiveness. Since $\mathrm{d}\gamma_S / \mathrm{d}\theta < 0$, it follows that the market competitiveness decreases in $\theta$.

## A.4. Proof of Proposition 2

From equation (A-11) and $\mathrm{E}[(i_n - i_{-n})^2] = \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 (1-\hat{\theta})^2 + (1-\rho)\left(1+\frac{1}{N-1}\right)$. It follows that the expected profit under objective belief is

$$\text{(A-12)} \qquad \mathrm{E}[\mathrm{Profit}] := \mathrm{E}\Big[(\mathrm{E}^n[v] - P)x_n\Big] = \frac{\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2 (1-\hat{\theta})^2 + (1-\rho)\left(1+\frac{1}{N-1}\right)}{\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2 (1-\theta)^2 + (1-\rho)\left(1+\frac{1}{N-1}\right)} \mathrm{E}^n[\mathrm{Profit}].$$

Equation (A-12) implies that if traders underestimate market crowdedness ($\theta < \hat{\theta}$), then $\mathrm{E}^n[\mathrm{Profit}] > \mathrm{E}[\mathrm{Profit}]$; If traders overestimate market crowdedness ($\theta > \hat{\theta}$), then $\mathrm{E}^n[\mathrm{Profit}] < \mathrm{E}[\mathrm{Profit}]$.

## A.5. Proof of Proposition 3

Direction calculation shows that $\frac{\mathrm{d}\lambda}{\mathrm{d}N_H} > 0$, and $\frac{\mathrm{d}\kappa}{\mathrm{d}N_H} > 0$. From equations (20) and (12), it follows that if traders overestimate $N_H$ while keeping $N_L$ fixed, then $N > \hat{N}$ and $1/\hat{\lambda} < 1/\lambda$. If traders underestimate $N_H$ while keeping $N_L$ fixed, then $N < \hat{N}$ and $1/\hat{\lambda} > 1/\lambda$.

## A.6. Proof of Lemma 2

Traders in the market believe that there are $N$ traders in the market and that noise in increments of their private signals are pairwise positively correlated with correlation coefficient $\rho$.

Stratonovich–Kalman–Bucy filtering implies that trader $n$'s estimate of the growth rate $G_n(t)$ satisfies the Itô differential equation

$$\begin{aligned} \mathrm{d}G_n(t) = &-\alpha_G\, G_n(t)\ \mathrm{d}t + \tau_0^{1/2}\sigma_G\, \Omega^{1/2}\Big(\mathrm{d}I_0(t) - \frac{\tau_0^{1/2}\,\mathrm{d}t}{\sigma_G\,\Omega^{1/2}} G_n(t)\Big) \\ &+ \frac{\left((1+(N-2)\rho)\tau_H^{1/2} - (N-1)\rho\tau_L^{1/2}\right)\sigma_G\,\Omega^{1/2}}{(1-\rho)(1+(N-1)\rho)}\Big(\mathrm{d}I_n(t) - \frac{\tau_H^{1/2}\,\mathrm{d}t}{\sigma_G\,\Omega^{1/2}} G_n(t)\Big) \\ &+ \frac{(\tau_L^{1/2} - \rho\tau_H^{1/2})\sigma_G\,\Omega^{1/2}}{(1-\rho)(1+(N-1)\rho)}\Big(\sum_{\substack{m=1\\ m\neq n}}^{N} \mathrm{d}I_m(t) - \frac{(N-1)\tau_L^{1/2}\,\mathrm{d}t}{\sigma_G\,\Omega^{1/2}} G_n(t)\Big). \end{aligned}$$

Rearranging terms yields

$$\begin{aligned}\mathrm{d}G_n(t) = &-(\alpha_G+\tau)\, G_n(t)\, \mathrm{d}t + \tau_0^{1/2}\sigma_G\,\Omega^{1/2}\,\mathrm{d}I_0(t) + \frac{(\tau_L^{1/2}-\rho\tau_H^{1/2})\sigma_G\,\Omega^{1/2}}{(1-\rho)(1+(N-1)\rho)} \sum_{\substack{m=1\\ m\neq n}}^{N} \mathrm{d}I_m(t) \\ &+ \frac{\left((1+(N-2)\rho)\tau_H^{1/2}-(N-1)\rho\tau_L^{1/2}\right)\sigma_G\,\Omega^{1/2}}{(1-\rho)(1+(N-1)\rho)}\ \mathrm{d}I_n(t),\end{aligned} \tag{A-13}$$

where the total precision of signals $\tau$ is as defined in Lemma 2 and the mean-square filtering error of the estimate $G(t)$, given by $\sigma_G^2\,\Omega(t)$ from equation (25), satisfies the Riccati differential equation,

$$\sigma_G^2\frac{\mathrm{d}\Omega(t)}{\mathrm{d}t} = -2\alpha_G\,\sigma_G^2\Omega(t) + \sigma_G^2 - \sigma_G^2\Omega(t)\left(\tau_0+\tau_H+(N-1)\frac{\left((\theta-\rho)\tau_H^{1/2}+(1-\theta)\tau_L^{1/2}\right)^2}{(1-\rho)(1+(N-1)\rho)}\right).$$

Let $\Omega$ denote the steady state of the function of time $\Omega(t)$. Using the steady-state assumption $\mathrm{d}\Omega(t)/\mathrm{d}t = 0$, we solve the Ricatti equation for the steady-state scaled error variance $\Omega = \Omega(t)$ to obtain equation (25), $\Omega := \mathrm{Var}^n\left[(G^*(t)-G_n(t))/\sigma_G\right] = (2\,\alpha_G+\tau)^{-1}$. The error variance $\Omega$ corresponds to a steady state that balances an increase in error variance due to innovations $\mathrm{d}B_G(t)$ in the true growth rate $G^*(t)$ with a reduction in error variance due to (1) mean reversion of the true growth rate at rate $\alpha_G$ and (2) the arrival of new information with total precision $\tau$. Note that $\Omega$ is not a free parameter, but is instead determined as an endogenous function of the other parameters. Equation (25) implies that $\Omega$ is the solution to the quadratic equation $\Omega^{-1} = 2\,\alpha_G+\tau$. In equations (22) and (23), we scaled the units with which precision is measured by the endogenous parameter $\Omega$. This leads to simpler filtering expressions. Define the processes $\mathrm{d}B_0^n$, $\mathrm{d}B_n^n$, and $\mathrm{d}B_m^n$, $m=1,\ldots,N$, $m \neq n$, by

$$\mathrm{d}B_0^n(t) = \tau_0^{1/2} \, \tfrac{G^*(t)-G_n(t)}{\sigma_G \, \Omega^{1/2}} \, \mathrm{d}t + \mathrm{d}B_0(t),$$

$$\mathrm{d}B_n^n(t) = \tau_H^{1/2} \, \tfrac{G^*(t)-G_n(t)}{\sigma_G \, \Omega^{1/2}} \, \mathrm{d}t + \rho^{1/2} \, \mathrm{d}Z(t) + (1-\rho)^{1/2} \, \mathrm{d}B_n(t),$$

$$\mathrm{d}B_m^n(t) = (\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}) \, \tfrac{G^*(t)-G_n(t)}{\sigma_G \, \Omega^{1/2}} \, \mathrm{d}t + \rho^{1/2} \, \mathrm{d}Z(t) + (1-\rho)^{1/2} \, \mathrm{d}B_m(t).$$

The superscript $n$ indicates conditioning on the beliefs of trader $n$. These $N+1$ processes are correlated distributed Brownian motions from the perspective of trader $n$. Trader $n$ believes that signals change as follows:

$$\mathrm{d}H_0(t) = -(\alpha_G + \tau) \, H_0(t) \, \mathrm{d}t + \tau_0^{1/2} \, \tfrac{G_n(t)}{\sigma_G \, \Omega^{1/2}} \, \mathrm{d}t + \mathrm{d}B_0^n(t),$$

$$\text{(A-14)} \qquad \mathrm{d}H_n(t) = -(\alpha_G + \tau) \, H_n(t) \, \mathrm{d}t + \tau_H^{1/2} \, \tfrac{G_n(t)}{\sigma_G \, \Omega^{1/2}} \, \mathrm{d}t + \mathrm{d}B_n^n(t),$$

$$\mathrm{d}H_{-n}(t) = -(\alpha_G + \tau) \, H_{-n}(t) \, \mathrm{d}t + (\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}) \, \tfrac{G_n(t)}{\sigma_G \, \Omega^{1/2}} \, \mathrm{d}t + \tfrac{1}{N-1} \sum_{\substack{m=1 \\ m \neq n}}^{N} \mathrm{d}B_m^n(t).$$

Note that each signal drifts toward zero at rate $\alpha_G + \tau$ and drifts toward the optimal forecast $G_n(t)$ at a rate proportional to the square root of the signal's precision $\tau_0^{1/2}$, $\tau_H^{1/2}$, or $\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}$, respectively. The estimate $G_n(t)$ can be conveniently written as the weighted sum of $N+1$ sufficient statistics $H_n(t)$ corresponding to $N+1$ information flows $\mathrm{d}I_n$. The sufficient statistics $H_n(t)$ is defined by equation (26). The trader $n$'s growth rate estimate $G_n(t)$ becomes a linear combination of sufficient statistics $H_n(t)$ as given by equation (27).

## A.7. Proof of Theorem 2

To reduce the number of state variables, it is convenient to replace the three state variables $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$ with two composite state variables $\breve{H}_n(t)$ and $\breve{H}_{-n}(t)$ defined by

$$\breve{H}_n(t) := H_n(t) + \breve{a} \, H_0(t), \quad \breve{H}_{-n}(t) := H_{-n}(t) + \breve{a} \, H_0(t), \qquad \text{where}$$

(A-15) $$\check{a} := \frac{(1+(N-1)\rho)\tau_0^{1/2}}{(1+(N-1)\theta)\tau_H^{1/2}+(N-1)(1-\theta)\tau_L^{1/2}}.$$

Trader $n$'s estimate of the dividend growth rate can be expressed as

$$G_n(t) := \sigma_G\, \Omega^{1/2} \left(\check{\tau}_H^{1/2}\, \check{H}_n(t) + (N-1)\check{\tau}_L^{1/2}\, \check{H}_{-n}(t)\right),$$

where $\check{\tau}_H^{1/2}$ and $\check{\tau}_L^{1/2}$ are defined as

(A-16) $$\check{\tau}_H^{1/2} := \frac{(1-\theta)(\tau_H^{1/2}-\tau_L^{1/2})}{1-\rho} + \frac{(\theta-\rho)\tau_H^{1/2}+(1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}, \qquad \check{\tau}_L^{1/2} := \frac{(\theta-\rho)\tau_H^{1/2}+(1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}.$$

Each trader $n$ exercises monopoly power in choosing how fast to demand liquidity from other traders to profit from his private information. He also exercises monopoly power in choosing how fast to provide liquidity to the other $N-1$ traders. Trader $n$ conjectures that the symmetric linear demand schedules for other traders $m$ ($m = 1, \ldots, N$, and $m \neq n$) is given by

(A-17) $$x_m(t) = \frac{\mathrm{d}S_m(t)}{\mathrm{d}t} = \gamma_D\, D(t) + \gamma_H\, \check{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t).$$

In his view, his flow-demand $x_n(t) = \mathrm{d}S_n(t)/\,\mathrm{d}t$ must satisfy the market clearing condition,

(A-18) $$x_n(t) + \sum_{\substack{m=1 \\ m\neq n}}^{N} \left(\gamma_D\, D(t) + \gamma_H\, \check{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t)\right) = 0,$$

which depends on his belief about the total number of traders in the market $N$. Using zero net supply restriction $\sum_{m=1}^{N} S_m(t) = 0$, he solves this equation for $P(t)$ as a function of his own trading speed $x_n(t)$ and infers the residual supply schedule that he implicitly faces,

(A-19) $$P(x_n(t)) = \frac{\gamma_D}{\gamma_P}\, D(t) + \frac{\gamma_H}{\gamma_P}\, \check{H}_{-n}(t) + \frac{\gamma_S}{(N-1)\gamma_P}\, S_n(t) + \frac{1}{(N-1)\gamma_P}\, x_n(t).$$

The inverse slopes with respect to $S_n(t)$ and $x_n(t)$ are the two measures of market depth $1/\lambda$ and $1/\kappa$, respectively. Then, trader $n$ solves for his optimal consumption and trading strategy by plugging the price impact function (A-19) into his dynamic optimization problem (24). Each trader believes that equilibrium prices reveal the average of private signals, and thus he implements this strategy by conditioning his trading speed directly on market prices. We conjecture that a steady-state value function has the form $V(M_n, S_n, D, \check{H}_n, \check{H}_{-n})$,

(A-20) $$V\left(M_n, S_n, D, \check{H}_n, \check{H}_{-n}\right) := \max_{[c_n(t), x_n(t)]} \mathrm{E}_t^n\left[\int_{s=t}^{\infty} -\mathrm{e}^{-\beta(s-t)-A\,c_n(s)}\,\mathrm{d}s\right],$$

where $P(x_n(t))$ is given by equation (A-19). Inventories follow $\mathrm{d}S_n(t) = x_n(t)\,\mathrm{d}t$, the change in cash holdings $\mathrm{d}M_n(t)$ follows $\mathrm{d}M_n(t) = (r\,M_n(t) + S_n(t)\,D(t) - c_n(t) - P(x_n(t))\,x_n(t))\,\mathrm{d}t$, and signals $\check{H}_n$ and $\check{H}_{-n}$ are given by

(A-21)
$$\mathrm{d}\check{H}_n(t) = -(\alpha_G + \tau)\,\check{H}_n(t)\,\mathrm{d}t + \frac{\tau_H^{1/2} + \check{a}\tau_0^{1/2}}{\sigma_G\,\Omega^{1/2}}\,G_n(t)\,\mathrm{d}t + \check{a}\,\mathrm{d}B_0^n(t) + \mathrm{d}B_n^n(t),$$
$$\mathrm{d}\check{H}_{-n}(t) = -(\alpha_G + \tau)\,\check{H}_{-n}(t)\,\mathrm{d}t + \frac{\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2} + \check{a}\tau_0^{1/2}}{\sigma_G\,\Omega^{1/2}}\,G_n(t)\,\mathrm{d}t + \check{a}\,\mathrm{d}B_0^n(t) + \frac{1}{N-1}\sum_{\substack{m=1\\ m\neq n}}^{N}\mathrm{d}B_m^n(t).$$

The dynamics of $\check{H}_n$ and $\check{H}_{-n}$ in equation (A-21) can be derived from equation (A-14). We conjecture and verify that the value function $V(M_n, S_n, D, \check{H}_n, \check{H}_{-n})$ has the specific form

(A-22)
$$V\left(M_n, S_n, D, \check{H}_n, \check{H}_{-n}\right) = -\exp\Big(\psi_0 + \psi_M M_n + \tfrac{1}{2}\psi_{SS}S_n^2 + \psi_{SD}S_n D$$
$$+ \psi_{Sn}\,S_n\check{H}_n + \psi_{Sx}\,S_n\check{H}_{-n} + \tfrac{1}{2}\psi_{nn}\,\check{H}_n^2 + \tfrac{1}{2}\psi_{xx}\,\check{H}_{-n}^2 + \psi_{nx}\,\check{H}_n\check{H}_{-n}\Big).$$

The nine constants $\psi_0$, $\psi_M$, $\psi_{SS}$, $\psi_{SD}$, $\psi_{Sn}$, $\psi_{Sx}$, $\psi_{nn}$, $\psi_{xx}$, and $\psi_{nx}$ have values consistent with a steady-state equilibrium. The Hamilton–Jacobi–Bellman (HJB) equation corresponding to the conjectured value function $V(M_n, S_n, D, \breve{H}_n, \breve{H}_{-n})$ is

$$
\begin{aligned}
0 = \max_{c_n, x_n} & \left[U(c_n) - \beta V + \tfrac{\partial V}{\partial M_n}\left(r M_n + S_n D - c_n - P(x_n)\, x_n\right) + \tfrac{\partial V}{\partial S_n} x_n\right] \\
& + \tfrac{\partial V}{\partial D}\left(-\alpha_D D + G_n(t)\right) + \tfrac{\partial V}{\partial \breve{H}_n}\left(-(\alpha_G + \tau)\breve{H}_n(t) + \tfrac{\tau_H^{1/2} + \breve{a}\tau_0^{1/2}}{\sigma_G\, \Omega^{1/2}} G_n(t)\right) + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial D^2}\sigma_D^2 \\
& + \tfrac{\partial V}{\partial \breve{H}_{-n}}\left(-(\alpha_G + \tau)\breve{H}_{-n}(t) + \tfrac{\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2} + \breve{a}\tau_0^{1/2}}{\sigma_G\, \Omega^{1/2}} G_n(t)\right) + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial \breve{H}_n^2}\left(1 + \breve{a}^2\right) \\
& + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial \breve{H}_{-n}^2}\left(\rho + \frac{1-\rho}{N-1} + \breve{a}^2\right) + \left(\tfrac{\partial^2 V}{\partial D \partial \breve{H}_n} + \tfrac{\partial^2 V}{\partial D \partial \breve{H}_{-n}}\right)\breve{a}\sigma_D + \tfrac{\partial^2 V}{\partial \breve{H}_n \partial \breve{H}_{-n}}(\rho + \breve{a}^2).
\end{aligned}
\tag{A-23}
$$

For the specific quadratic specification of the value function, the HJB equation becomes

$$
\begin{aligned}
0 = \min_{c_n, x_n} & \left[-\tfrac{1}{V}\, \mathrm{e}^{-A c_n} - \beta + \psi_M\left(r M_n + S_n\, D - c_n - P(x_n)\, x_n\right)\right. \\
& \qquad \left. + \left(\psi_{SS} S_n + \psi_{SD} D + \psi_{Sn}\breve{H}_n + \psi_{Sx}\breve{H}_{-n}\right)x_n\right] + \psi_{SD} S_n\left(-\alpha_D D + G_n(t)\right) \\
& + \left(\psi_{Sn} S_n + \psi_{nn}\breve{H}_n + \psi_{nx}\breve{H}_{-n}\right)\left(-(\alpha_G + \tau)\breve{H}_n(t) + \tfrac{\tau_H^{1/2} + \breve{a}\tau_0^{1/2}}{\sigma_G\, \Omega^{1/2}} G_n(t)\right) \\
& + \left(\psi_{Sx} S_n + \psi_{xx}\breve{H}_{-n} + \psi_{nx}\breve{H}_n\right)\left(-(\alpha_G + \tau)\breve{H}_{-n}(t) + \tfrac{\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2} + \breve{a}\tau_0^{1/2}}{\sigma_G\, \Omega^{1/2}} G_n(t)\right) \\
& + \tfrac{1}{2}\psi_{SD}^2 S_n^2 \sigma_D^2 + \tfrac{1}{2}\left(\left(\psi_{Sn} S_n + \psi_{nn}\breve{H}_n + \psi_{nx}\breve{H}_{-n}\right)^2 + \psi_{nn}\right)\left(1 + \breve{a}^2\right) \\
& + \tfrac{1}{2}\left(\left(\psi_{Sx} S_n + \psi_{xx}\breve{H}_{-n} + \psi_{nx}\breve{H}_n\right)^2 + \psi_{xx}\right)\left(\rho + \tfrac{1-\rho}{N-1} + \breve{a}^2\right) \\
& + \left((\psi_{Sn} + \psi_{Sx}) S_n + (\psi_{nn} + \psi_{nx})\breve{H}_n + (\psi_{xx} + \psi_{nx})\breve{H}_{-n}\right)\psi_{SD} S_n \breve{a}\sigma_D \\
& + \left(\left(\psi_{Sn} S_n + \psi_{nn}\breve{H}_n + \psi_{nx}\breve{H}_{-n}\right)\left(\psi_{Sx} S_n + \psi_{xx}\breve{H}_{-n} + \psi_{nx}\breve{H}_n\right) + \psi_{nx}\right)(\rho + \breve{a}^2).
\end{aligned}
$$

The solution for optimal consumption is

$$
c_n(t) = -\tfrac{1}{A}\, \log\left(\tfrac{\psi_M\, V(t)}{A}\right). \tag{A-24}
$$

The optimal trading strategy is a linear function of the state variables given by

$$
\begin{aligned}
x_n(t) = \tfrac{(N-1)\gamma_P}{2\psi_M} & \left(\left(\psi_{SD} - \tfrac{\psi_M \gamma_D}{\gamma_P}\right) D(t) + \left(\psi_{SS} - \tfrac{\psi_M \gamma_S}{(N-1)\gamma_P}\right) S_n(t)\right. \\
& \left. + \psi_{Sn}\, \breve{H}_n(t) + \left(\psi_{Sx} - \tfrac{\psi_M \gamma_H}{\gamma_P}\right) \breve{H}_{-n}(t)\right).
\end{aligned}
\tag{A-25}
$$

Trader $n$ can infer from the market-clearing condition (A-18) that $\breve{H}_{-n}$ is given by

$$\text{(A-26)} \qquad \breve{H}_{-n}(t) = \tfrac{\gamma_P}{\gamma_H} \left(P(t) - D(t)\, \tfrac{\gamma_D}{\gamma_P}\right) - \tfrac{1}{(N-1)\gamma_H}\, x_n(t) - \tfrac{\gamma_S}{(N-1)\gamma_H}\, S_n(t).$$

Plugging equation (A-26) into (A-25) yields $x_n(t)$ as a linear demand schedule given by

$$\text{(A-27)} \qquad \begin{aligned} x_n(t) = &\tfrac{(N-1)\gamma_P}{\psi_M} \left(1 + \tfrac{\psi_{Sx}}{\psi_M} \tfrac{\gamma_P}{\gamma_H}\right)^{-1} \\ &\cdot \left( \left(\psi_{SD} - \psi_{Sx} \tfrac{\gamma_D}{\gamma_H}\right) D(t) + \left(\psi_{SS} - \psi_{Sx} \tfrac{\gamma_S}{(N-1)\gamma_H}\right) S_n(t) \right. \\ &\left. + \psi_{Sn}\, \breve{H}_n(t) + \left(\psi_{Sx} \tfrac{\gamma_P}{\gamma_H} - \psi_M\right) P(t) \right). \end{aligned}$$

Equating the coefficients of $D(t)$, $\breve{H}_n(t)$, $S_n(t)$, and $P(t)$ in equation (A-27) to the conjectured coefficients $\gamma_D$, $\gamma_H$, $-\gamma_S$, and $-\gamma_P$ results in the following four equations:

$$\tfrac{(N-1)\gamma_P}{\psi_M} \left(1 + \tfrac{\psi_{Sx}}{\psi_M} \tfrac{\gamma_P}{\gamma_H}\right)^{-1} \left(\psi_{SD} - \psi_{Sx} \tfrac{\gamma_D}{\gamma_H}\right) = \gamma_D, \qquad \tfrac{(N-1)\gamma_P}{\psi_M} \left(1 + \tfrac{\psi_{Sx}}{\psi_M} \tfrac{\gamma_P}{\gamma_H}\right)^{-1} \psi_{Sn} = \gamma_H,$$

$$\tfrac{(N-1)\gamma_P}{\psi_M} \left(1 + \tfrac{\psi_{Sx}}{\psi_M} \tfrac{\gamma_P}{\gamma_H}\right)^{-1} \left(\psi_{SS} - \psi_{Sx} \tfrac{\gamma_S}{(N-1)\gamma_H}\right) = -\gamma_S, \qquad \tfrac{(N-1)\gamma_P}{\psi_M} \left(1 + \tfrac{\psi_{Sx}}{\psi_M} \tfrac{\gamma_P}{\gamma_H}\right)^{-1} \left(\psi_{Sx} \tfrac{\gamma_P}{\gamma_H} - \psi_M\right) = -\gamma_P.$$

We obtain

$$\text{(A-28)} \qquad \psi_{Sx} = \tfrac{N-2}{2} \psi_{Sn}, \qquad \gamma_H = \tfrac{N\gamma_P}{2\psi_M} \psi_{Sn}, \qquad \gamma_S = -\tfrac{(N-1)\gamma_P}{\psi_M} \psi_{SS}, \qquad \gamma_D = \tfrac{\gamma_P}{\psi_M} \psi_{SD}.$$

Define the constants $C_L$ and $C_G$ by

$$\text{(A-29)} \qquad C_L := -\tfrac{\psi_{Sn}}{2\psi_{SS}}, \qquad C_G := \tfrac{\psi_{Sn}}{2\psi_M}\, \frac{N(r+\alpha_D)(r+\alpha_G)\left(1+(N-1)\rho\right)}{\sigma_G \Omega^{1/2} \left((1+(N-1)\theta)\tau_H^{1/2} + (N-1)(1-\theta)\tau_L^{1/2}\right)}.$$

Substituting equation (A-28) into equation (A-25) yields the solution for optimal strategy.

$$x_n(t) = \gamma_S \left(C_L\, (H_n(t) - H_{-n}(t)) - S_n(t)\right).$$

The equilibrium price is

$$P(t) = \tfrac{D(t)}{r+\alpha_D} + \tfrac{C_G \sum_{n=1}^N G_n(t)}{N(r+\alpha_D)(r+\alpha_G)}.$$

Plugging (A-24) and (A-25) back into the Bellman equation and setting the constant term and the coefficients of $M_n$, $S_n\, D$, $S_n^2$, $S_n\, \check{H}_n$, $S_n\, \check{H}_{-n}$, $\check{H}_n^2$, $\check{H}_{-n}^2$, and $\check{H}_n\, \check{H}_{-n}$ to be zero, we obtain nine equations. Using the first equation (A-28) to substitute $\psi_{Sn}$ for $\psi_{Sx}$, there are in total nine equations in nine unknowns $\gamma_P$, $\psi_0, \psi_M, \psi_{SD}, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and $\psi_{nx}$. By setting the constant term, coefficient of $M$, and coefficient of $S_n D$ to be zero, we obtain

$$\psi_M = -rA, \qquad \psi_{SD} = -\tfrac{rA}{r+\alpha_D},$$

$$\psi_0 = 1 - \ln r + \tfrac{1}{r}\left(-\beta + \tfrac{1}{2}(1+\check{a}^2)\psi_{nn} + \tfrac{1}{2}\left(\check{a}^2 + \tfrac{1+(N-2)\rho}{N-1}\right)\psi_{xx} + (\check{a}^2+\rho)\psi_{nx}\right).$$

In addition, by setting the coefficients of $S_n^2, S_n\, \check{H}_n, S_n\, \check{H}_{-n}, \check{H}_n^2, \check{H}_{-n}^2$ and $\check{H}_n\, \check{H}_{-n}$ to be zero, we obtain six polynomial equations in the six unknowns $\gamma_P, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and $\psi_{nx}$. Defining the constants $a_1$, $a_2$, $a_3$, and $a_4$ by

(A-30)

$$a_1 := -\alpha_G - \tau + \check{\tau}_H^{1/2}(\tau_H^{1/2} + \check{a}\tau_0^{1/2}), \quad a_2 := -\alpha_G - \tau + (N-1)\check{\tau}_L^{1/2}(\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2} + \check{a}\tau_0^{1/2}),$$

$$a_3 := (\tau_H^{1/2} + \check{a}\tau_0^{1/2})(N-1)\check{\tau}_L^{1/2}, \qquad a_4 := (\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2} + \check{a}\tau_0^{1/2})\check{\tau}_H^{1/2},$$

these six equations in six unknowns can be written

(A-31)

$$0 = -\tfrac{1}{2}r\psi_{SS} - \tfrac{\gamma_P(N-1)}{rA}\psi_{SS}^2 + \tfrac{r^2A^2\sigma_D^2}{2(r+\alpha_D)^2} + \tfrac{1}{2}(1+\check{a}^2)\psi_{sn}^2 +$$
$$+ \tfrac{1}{2}\left(\check{a}^2 + \tfrac{1+(N-2)\rho}{N-1}\right)\tfrac{(N-2)^2}{4}\psi_{Sn}^2 - \tfrac{rA}{r+\alpha_D}\check{a}\sigma_D\tfrac{N}{2}\psi_{Sn} + \tfrac{N-2}{2}\psi_{Sn}^2(\check{a}^2+\rho),$$

(A-32)

$$0 = -r\psi_{Sn} - \tfrac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}\left(\tfrac{(1-\theta)(\tau_H^{1/2}-\tau_L^{1/2})}{1-\rho} + \tfrac{(\theta-\rho)\tau_H^{1/2}+(1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}\right) + a_1\psi_{Sn}$$
$$- \tfrac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn} + \tfrac{N-2}{2}a_4\psi_{Sn} + (1+\check{a}^2)\psi_{nn}\psi_{Sn} + \tfrac{N-2}{2}\left(\check{a}^2 + \tfrac{1+(N-2)\rho}{N-1}\right)\psi_{nx}\psi_{Sn}$$
$$- \tfrac{rA}{r+\alpha_D}\check{a}\sigma_D(\psi_{nn}+\psi_{nx}) + (\check{a}^2+\rho)\left(\psi_{nx}\psi_{Sn} + \tfrac{N-2}{2}\psi_{nn}\psi_{Sn}\right),$$

$$0 = -r\tfrac{N-2}{2}\psi_{Sn} + \tfrac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn} - \tfrac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}(N-1)\tfrac{(\theta-\rho)\tau_H^{1/2}+(1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}$$

(A-33) $$+ (a_3 + \tfrac{N-2}{2}a_2)\psi_{Sn} + (1+\breve{a}^2)\psi_{Sn}\psi_{nx} + \tfrac{N-2}{2}\left(\breve{a}^2 + \tfrac{1+(N-2)\rho}{N-1}\right)\psi_{xx}\psi_{Sn}$$

$$- \tfrac{rA}{r+\alpha_D}\breve{a}\sigma_D(\psi_{xx}+\psi_{nx}) + (\breve{a}^2+\rho)\left(\psi_{xx}\psi_{Sn} + \tfrac{N-2}{2}\psi_{nx}\psi_{Sn}\right),$$

(A-34)

$$0 = -\tfrac{r}{2}\psi_{nn} - \tfrac{\gamma_P(N-1)}{4rA}\psi_{Sn}^2 + a_1\psi_{nn} + a_4\psi_{nx} + \tfrac{1}{2}(1+\breve{a}^2)\psi_{nn}^2 + (\breve{a}^2+\rho)\psi_{nn}\psi_{nx}$$

$$+ \tfrac{1}{2}\left(\breve{a}^2 + \tfrac{1+(N-2)\rho}{N-1}\right)\psi_{nx}^2,$$

(A-35) $$0 = -\tfrac{r}{2}\psi_{xx} - \tfrac{\gamma_P(N-1)}{4rA}\psi_{Sn}^2 + a_2\psi_{xx} + a_3\psi_{nx} + \tfrac{1+\breve{a}^2}{2}\psi_{nx}^2 + (\breve{a}^2+\rho)\psi_{xx}\psi_{nx}$$

$$+ \tfrac{1}{2}\left(\breve{a}^2 + \tfrac{1+(N-2)\rho}{N-1}\right)\psi_{xx}^2,$$

(A-36)

$$0 = -r\psi_{nx} + \tfrac{\gamma_P(N-1)}{2rA}\psi_{Sn}^2 + a_3\psi_{nn} + a_4\psi_{xx} + (a_1+a_2)\psi_{nx} + (1+\breve{a}^2)\psi_{nn}\psi_{nx}$$

$$+ \left(\breve{a}^2 + \tfrac{1+(N-2)\rho}{N-1}\right)\psi_{xx}\psi_{nx} + (\breve{a}^2+\rho)(\psi_{nn}\psi_{xx} + \psi_{nx}^2).$$

Obtaining a solution in Theorem 2 requires solving the six polynomial equations (A-31)–(A-36). While these equations have no obvious analytical solution, they can be solved numerically. For a solution to the six polynomial equations to define a stationary equilibrium, a second-order condition implying $\gamma_P > 0$, a stationarity condition implying $\gamma_S > 0$, and a transversality condition requiring $r > 0$.

If risk aversion $A$ is scaled by a factor of $F$ to $A/F$, then $C_L$ changes to $C_L\,F$, $\lambda$ changes to $\lambda/F$, $\kappa$ changes to $\kappa/F$, $S_n^{TI}(t)$ changes to $S_n^{TI}(t)\,F$, while $\gamma_S$, $C_G$, and the price remain unchanged.

To see this, consider a vector $(\gamma_P^*, \psi_{SS}^*, \psi_{Sn}^*, \psi_{nn}^*, \psi_{nx}^*, \psi_{xx}^*)$ as a solution to the system (A-31)–(A-36) for the exogenous parameters $A$, $\sigma_D$, $\sigma_G$, $r$, $\alpha_G$, $\alpha_D$, $\tau_0$, $\tau_L$, and $\tau_H$. If risk aversion is rescaled by factor $F$ from $A$ to $A/F$ and other exogenous parameters remain unchanged, it can be shown that the vector $(\gamma_P^* F,\ \psi_{SS}^*/F^2,\ \psi_{Sn}^*/F,\ \psi_{nn}^*,\ \psi_{nx}^*,\ \psi_{xx}^*)$ is the solution to the system (A-31)–(A-36). From equations (A-28) and (A-29), it follows that $C_L$ changes to $C_L\,F$, $\lambda$ changes to $\lambda/F$, $\kappa$ changes to $\kappa/F$, but $\gamma_S$ and $C_G$ remain unchanged.

The transversality condition for the value function $V(\ldots)$ is

$$\lim_{T\to+\infty} \mathrm{E}_t^n\left[\mathrm{e}^{-\rho(T-t)}\ V\left(M_n(T), S_n(T), D(T), \check{H}_n(T), \check{H}_{-n}(T)\right)\right] = 0.$$

The transversality condition (A-37) is satisfied if $r > 0$. Under the assumptions $\gamma_P > 0$ and $\gamma_S > 0$, analytical results imply $\gamma_D > 0$, $\psi_M < 0$, $\psi_{SD} < 0$, and $\psi_{SS} > 0$. The numerical results indicate that $\gamma_H > 0$, $\psi_{Sn} < 0$, $\psi_{Sx} < 0$, $\psi_{nn} < 0$, $\psi_{xx} < 0$ and the sign of $\psi_{nx}$ is intuitively and numerically ambiguous.

# Internet Appendix for "Trading in Crowded Markets"

Section B.1 presents an alternative one-period model without disagreement. Section B.2 presents continuous-time model implications. Section B.3 discusses whether traders can learn about their mistakes in assessing the degree of market crowdedness from observing dynamics of market prices. Section B.4 presents the equilibrium of the continuous-time model with perfect competition. Section B.5 provides details on return autocorrelations.

## B.1. An Alternative Model of Crowded Markets

In this section, we consider an alternative one-period model, in which we replace disagreement about signal precisions (model with different priors) with private values (model with a common prior) as the modeling device to generate trade. We show that this alternative modelling is much less appealing for studying crowded markets. Specifically, as the correlation of private values rises, the correlation of traders' composite signals increases but traders are willing to provide more (not less) liquidity to one another, and as a result, they trade more aggressively and the market becomes more liquid.

Similar to the one-period model with different priors, there are $N$ strategic traders who trade a risky asset with a random payoff $v$, $v \sim N(0, 1/\tau_v)$, against a riskless asset. Each trader $n$ observes a private signal $i_n$ of precision $\tau_I$ with a noise term, $i_n := \tau_I^{1/2} (\tau_v^{1/2} v) + e_n$, where $e_n \sim N(0,1)$, $n = 1, 2, \ldots, N$, and $v$, $e_1, \ldots, e_N$ are independently distributed. Since traders agree on how much information each signal contains, the traders share a common prior.

Unlike in the model with disagreement, the risky asset generates privately observed private benefits for traders owning it, and this helps to generate trade. Assume that trader $n$'s cash-equivalent of the private benefit from holding the risky asset is $\beta h_n$, where $h_n = \rho_h^{1/2} z + (1-\rho_h)^{1/2} e_{hn}$. In the expression of $h_n$, $z \sim N(0,1)$, $e_{hn} \sim N(0,1)$, $n = 1,2,\ldots,N$, and $v$, $z$, $e_{h1},\ldots,e_{hN}$ are independently distributed. Traders' private values are correlated with correlation coefficient of $\rho_h$. Thus, the risky asset generates a cash flow $v + \beta\, h_n$ for trader $n$, where the additional component $\beta h_n$ is a privately observed "convenience yield," independent of $v$.

We show that each trader can infer from the equilibrium price the average of a linear combination of other traders' private information and private values, $\sum_{m=1,m\neq n}^{N}(i_m + kh_m)$. The private values essentially show up as noise in prices. The value of the weight $k$ on private values is determined endogenously in equilibrium.

Denote the linear combination of trader $n$'s private information and private value as $I_n$ and the average of other traders' private information and private values as $I_{-n}$,

(B-1) $$I_n := i_n + kh_n, \qquad I_{-n} := \tfrac{1}{N-1}\sum_{m=1,m\neq n}^{N}(i_m + kh_m).$$

Upon observing signals $i_n$ and given his current inventory $S_n$, each trader $n$ submits a demand schedule $X_n(P) := X_n(i_n, h_n, S_n, P)$ as a function of price $P$ to a single-price auctioneer who clears the market at price $P$. He chooses an optimal demand schedule to maximize the expected negative exponential utility function with risk aversion $A$, as defined in equation (5). Trader $n$'s terminal wealth is given as $W_n := (v + \beta h_n)\,(S_n + X_n(P)) - P\,X_n(P)$, where $v + \beta h_n$ measures the cash-equivalent of the private benefit of owning the asset. As in our main model, each trader $n$ strategically exercises his market power by taking into account that his trading quantity $x_n$ directly affects the price $P(x_n)$ on the residual supply schedule he faces.

Conjecturing that other traders submit symmetric linear demand schedules,

(B-2) $$X_m(i_0, i_m, S_m, P) = \gamma_I\,(i_m + kh_m) - \gamma_P\,P - \gamma_S\,S_m, \qquad m = 1, \ldots, N, \quad m \neq n,$$

and knowing the market-clearing condition holds in equilibrium,

$$x_n + \sum_{\substack{m=1 \\ m\neq n}}^{N} (\gamma_I\,(i_m + kh_m) - \gamma_P\,P - \gamma_S\,S_m) = 0,$$

each trader $n$ infers the residual supply schedule $P(x_n)$ he faces,

(B-3) $$P(x_n) = \frac{\gamma_I}{\gamma_P}\,I_{-n} + \frac{\gamma_S}{(N-1)\gamma_P}\,S_n + \frac{1}{(N-1)\gamma_P}\,x_n.$$

The two measures of market depth are $1/\lambda := (N-1)\,\gamma_P/\gamma_S$ and $1/\kappa := (N-1)\,\gamma_P$. The following theorem characterizes the equilibrium.

**Theorem B.1. One-Period Model with Private Values.** *The endogenous weight $k$ on private values is the unique real root to the cubic equation,*

(B-4) $$k\tau_I^{1/2} = \beta\tau_v^{1/2}\Big(1 + \tau_I + \frac{(N-1)\tau_I}{1 + k^2(1 + (N-2)\rho_h)}\Big).$$

*If the second-order condition, $k^2 > N\left((N-2)((N-2)\rho_h + 1)\right)^{-1}$ holds, an equilibrium exists:*

*1. The parameters $\gamma_S > 0$, $\gamma_I > 0$, and $\gamma_P > 0$, defining the linear trading strategies in equation (B-2), have unique closed-form solutions presented in equation (B-8) in the Appendix.*

2. *Trader $n$ trades the quantity $x_n$ towards his target inventory $S_n^{TI}$,*

(B-5) $$x_n = \gamma_S \,(S_n^{TI} - S_n), \qquad S_n^{TI} = \frac{(N-1)k^2(1+(N-2)\rho_h)\tau_I^{1/2}\tau_v^{1/2}}{AN(1+k^2(1+(N-2)\rho_h))}\,(I_n - I_{-n}),$$

*where $0 < \gamma_S < 1$, is the fraction by which trader $n$ adjusts his position $S_n$.*

3. *The equilibrium price $P$ is proportional to the average of traders' composite signals $\sum_{n=1}^{N} I_n$,*

(B-6) $$P = \frac{(N+k^2(1+(N-2)\rho_h))\tau_I^{1/2}\tau_v^{1/2}}{N\;\tau\;(1+k^2((N-2)\rho_h+1))}\sum_{n=1}^{N} I_n,$$

*where the "total precision" is given by $\tau := (\mathrm{Var}^n[v])^{-1}$.*

**Proof of Theorem B.1:**

Each trader updates his estimates of the expectation and the variance of fundamental value $v$, denoted as $\mathrm{E}^n[v]$ and $\mathrm{Var}^n[v]$, respectively. Define "total precision" $\tau$ as the inverse of posterior variance $\tau := (\mathrm{Var}^n[v])^{-1}$. Each trader $n$ forms the following estimates of posterior expectation and variance,

(B-7)

$$\tau := (\mathrm{Var}^n[v])^{-1} = \tau_v\left(1 + \tau_I + \frac{(N-1)\tau_I}{1+k^2((N-2)\rho_h+1)}\right), \qquad \mathrm{E}^n[v] = \frac{\tau_v^{1/2}\tau_I^{1/2}}{\tau}\left(i_n + \frac{N-1}{1+k^2((N-2)\rho_h+1)} I_{-n}\right).$$

The proof of Theorem B.1 is similar to the proof of Theorem 1. Equating the corresponding coefficients of the variables $I_n$, $P$, and $S_n$ yields a system of three equations in terms of the three unknowns $\gamma_S$, $\gamma_I$, and $\gamma_P$. The second-order condition has the correct sign if and only if $2((N-1)\gamma_P)^{-1} + A\mathrm{Var}^n[v] > 0$. This is equivalent to

$k^2 > N((N-2)((N-2)\rho_h+1))^{-1}$. The unique solution of $\gamma_S$, $\gamma_I$, and $\gamma_P$ are

$$\gamma_S := \frac{N-2}{N-1} - \frac{N}{k^2(N-1)(1+(N-2)\rho_h)}, \qquad \gamma_I := \frac{(-N+k^2(N-2)(1+(N-2)\rho_h))\tau_I^{1/2}\tau_v^{1/2}}{A(N-1)(1+k^2(1+(N-2)\rho_h))},$$
$$\gamma_P := \frac{(1+N\tau_I+k^2(1+(N-2)\rho_h)(1+\tau_I))\tau_v^{1/2}}{(N+k^2(1+(N-2)\rho_h))\tau_I^{1/2}}\gamma_I. \tag{B-8}$$

As in our main model, we can potentially define market crowdedness as the correlation of traders' composite signals $\text{Corr}^n[I_n, I_{-n}]$. Extensive numerical analysis suggests that this correlation increases in the correlation of private values, $\rho_h$, as shown in the left panel of Figure B-1.[1] In the right panel of Figure B-1, we vary $\rho_h$ and depict the model's implications about how trade aggressiveness $\gamma_S$ changes with market crowdedness. Figure B-2 shows model's implications for how market depth $1/\lambda$ and $1/\kappa$ changes with market crowdedness. The model implies that when the market becomes more crowded, traders trade more aggressively, market depth increases and market becomes more liquid.[2]

These results are exactly the opposite to those of our main model with agreement to disagree, in which traders trade less aggressively and markets become less liquid, as crowdedness goes up. In the setting where traders trade due to private values, private values are essentially not related to fundamentals. When the correlation of private values goes up, the correlation of signals increases but traders are willing to provide more (not less) liquidity to one another, and as a result, they trade more aggressively and the market becomes more liquid. Similar results would be observed in the model of Section A, if one increases the correlation in noise term of signals ($\rho$). It is therefore important whether the correlation of signals increases due to the correlation in noise

[1] In both Figures B-1 and B-2, the default parameter values are $\tau_I = 0.2$, $\beta = 2$, $\tau_v = 1$, $N = 40$, and $A = 1$.

[2] If the correlation of signals ($\text{Corr}^n[I_n, I_{-n}]$) increases as a result of a higher $N$ or a higher $\tau_I$, extensive numerical analysis also indicates that traders tend to engage in more aggressive trading, leading to increased market liquidity.

FIGURE B-1

**$\text{Corr}^n[I_n, I_{-n}]$ against $\rho_h$ and $\gamma_S$ against $\rho_h$**

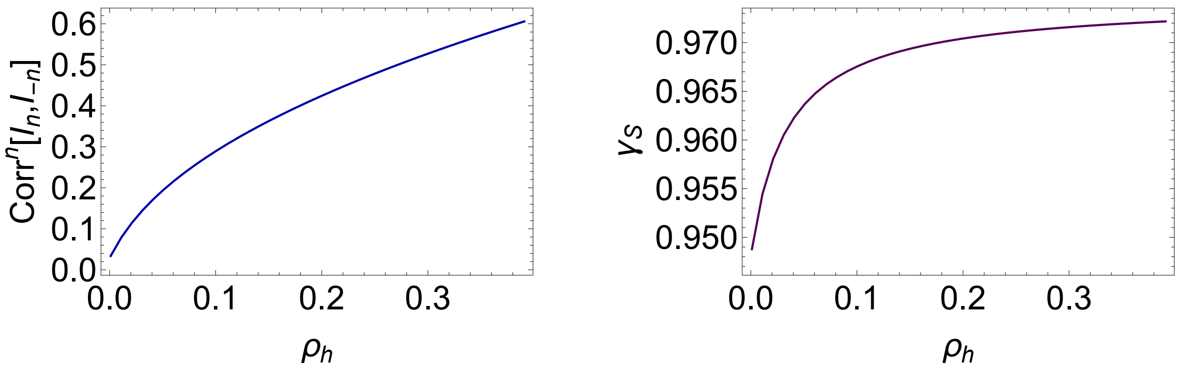
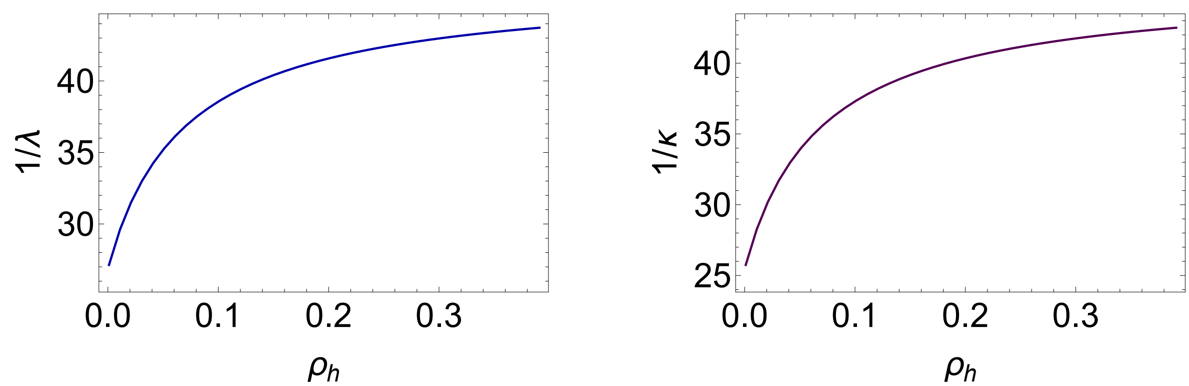



FIGURE B-2

**Values of $1/\lambda$ and $1/\kappa$ against $\rho_h$**




terms or the correlation in information content of signals, because these assumptions lead to drastically different implications. Since our goal is to model crowdedness among sophisticated traders, we chose the model where traders agree to disagree about interpretation of their private signals and study effects of market crowdedness by essentially varying the correlation in signals' information content.

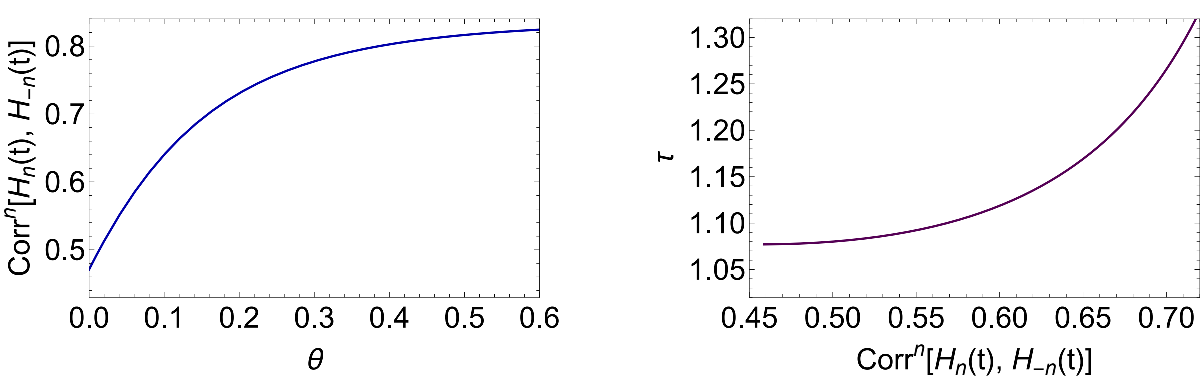
## B.2. Continuous-time Model Implications

This section shows that when traders perceive a high proportion of others observing high-precision signals, they trade less aggressively and target smaller positions. Conversely, a lower perceived proportion of traders with high-precision signals leads to more aggressive trading.

Figure B-3 demonstrates numerically that the perceived correlation of private signals increases with the fraction of traders whose signal are of high precision.[3] As in the one-period model, a higher value of $\theta$ results in a higher correlation of private signals. Additionally, the correlation of private signals is positively associated with the total amount of information, $\tau$, as long as the correlation of noise terms of their private signals is not large. Intuitively, more traders with high-precision signals bring more information to the market through their trading.

FIGURE B-3

**$\mathrm{Corr}^n[H_n(t), H_{-n}(t)]$ against $\theta$ and $\tau$ against $\mathrm{Corr}^n[H_n(t), H_{-n}(t)]$**




The left panel of Figure B-4 plots the speed of trading $\gamma_S$ against $\theta$.[4] When traders perceive the market as highly crowded, they trade less aggressively, leading to a decrease in $\gamma_S$.

[3] In Figure B-3, parameter values $\beta = 0.05$ and $\tau_H = 1$. Parameter values $r$, $A$, $\alpha_D$, $\alpha_G$, $\sigma_D$, $\sigma_G$, $N$, $\rho$, and $\tau_L$ are presented in Table 1 in Section A.

[4] In Figures B-4 and B-5, $\tau_H = 1$ and parameter values $r$, $A$, $\alpha_D$, $\alpha_G$, $\sigma_D$, $\sigma_G$, $N$, $\rho$, and $\tau_L$ are presented in Table 1 in Section A. We vary $N_H$ with $N$ fixed.

The right panel of Figure B-4 demonstrates that traders aim for smaller positions under the belief of a crowded market, as the perceived profit diminishes.

FIGURE B-4

**Values of $\gamma_S$ and $E^n|S_n^{TI}(t)|$ against $\theta$ by varying $N_H$ with $N$ fixed**

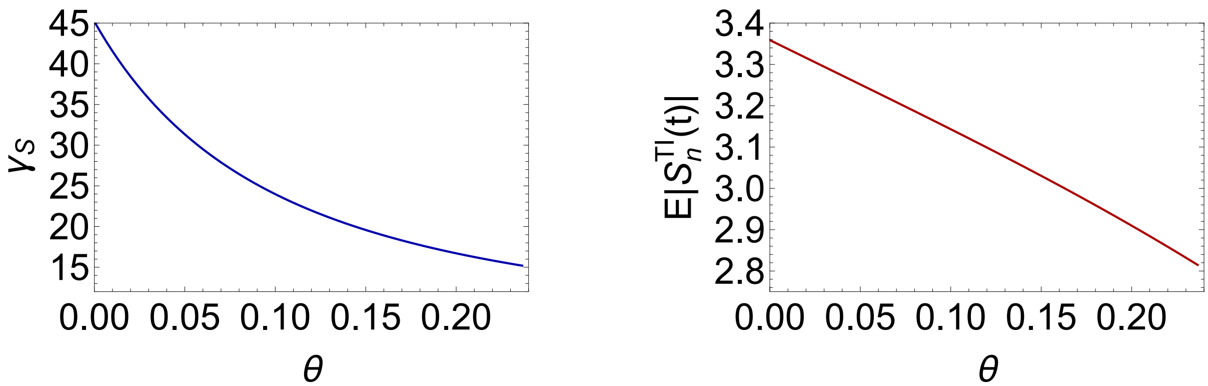
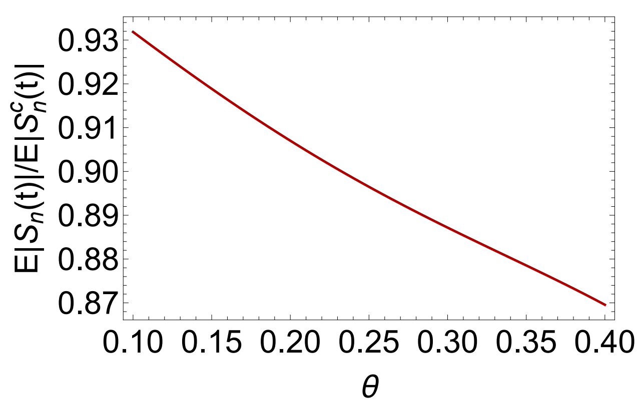



FIGURE B-5

**Market competitiveness against $\theta$ by varying $N_H$ with $N$ fixed**

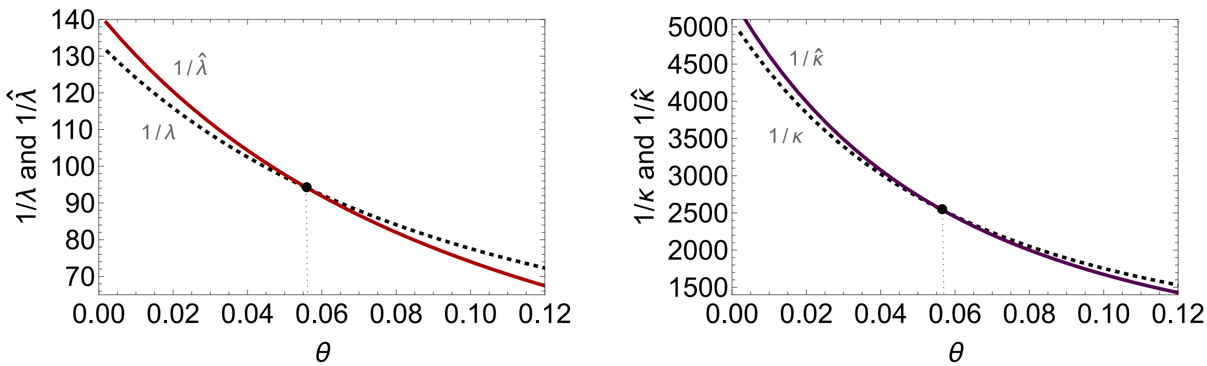



Section B.4 characterizes the equilibrium of an otherwise equivalent continuous-time model, where the assumption of perfect competition replaces imperfect competition. Figure B-5 shows that market competitiveness decreases as the proportion of high-precision signals increases, since each trader becomes less willing to provide liquidity to others.

FIGURE B-6

**Values of $1/\hat{\lambda}$, $1/\lambda$, $1/\hat{\kappa}$, and $1/\kappa$**

This figure presents values of $1/\hat{\lambda}$, $1/\lambda$, $1/\hat{\kappa}$, and $1/\kappa$ against perceived market crowdedness $\theta$ by varying $N_H$ with $N_L = \hat{N}_L$ fixed. The dashed curves correspond to subjective liquidity metrics $1/\lambda$ and $1/\kappa$. The solid curves correspond to objective metrics $1/\hat{\lambda}$ and $1/\hat{\kappa}$.




**Perceived versus Actual Market Depth.** The left panel of Figure B-6 plots the permanent market depth $1/\hat{\lambda}$ (solid curve) and its subjective estimate $1/\lambda$ (dashed curve) against perceived market crowdedness $\theta$.[5] The right panel of Figure B-6 presents similar plots for the temporary market depth $1/\hat{\kappa}$ (solid curve) and its subjective estimate $1/\kappa$ (dashed curve). Similar to the one-period model results, when traders overestimate the crowdedness of the market (to the right of the benchmark point), they expect that the market depth is low, because everybody is less willing to provide liquidity to each other. In reality, the actual market depth is even lower than what traders think, $1/\hat{\lambda} < 1/\lambda$ and $1/\hat{\kappa} < 1/\kappa$. In contrast, when traders underestimate the crowdedness of the market (to the left of the benchmark point), all types of market depth increase, because traders are more aggressive in trading on private information and providing liquidity to others. The actual market depth is even higher than the perceived one ($1/\hat{\lambda} > 1/\lambda$ and $1/\hat{\kappa} > 1/\kappa$). Traders could learn about the actual residual supply schedule's slopes by implementing a series of

[5] In Figure B-6, parameter values $r$, $A$, $\alpha_D$, $\alpha_G$, $\sigma_D$, $\sigma_G$, $\rho$, and $\tau_L$ are presented in Table 1 in Section A, $\tau_H = 1$, and $\hat{N}_H = 4$. We vary $N_H$ with $N_L = \hat{N}_L = 50$ fixed.

experiments and analyzing price responses to trading at some off-equilibrium trading rates. Even if traders could learn about the number of traders ($\hat{N}$) by obtaining some data on residual supply schedules, they still cannot learn in real time exactly how many traders ($\hat{N}_H$) are trading with them in the same direction. Therefore, we mainly focus on the case of varying $N_H$ with $N = \hat{N}$ fixed.

## B.3. Can Traders Learn about Their Mistakes for Price Dynamics?

Let $\hat{N} := \hat{N}_H + \hat{N}_L$ denote the actual total number of traders, out of whom $\hat{N}_H$ traders observe high precision signals and $\hat{N}_L$ traders observe low precision signals.

It is natural to ask whether traders can learn about their mistakes in assessing the degree of market crowdedness from observing dynamics of market prices.

Each trader observes the market price $\hat{P}(t)$ but thinks of it as his conjectured price $P(t)$. Each trader infers from the price the average of $\hat{N}$ signals, but interprets it as the average of $N$ signals. Since in the continuous time model it is possible to accurately estimate the diffusion variance of the process $P(t)$ by looking at its quadratic variation, one may think that traders would be able to learn about their mistakes from the price dynamics. For example, if traders underestimated the total number of participants in the market ($N < \hat{N}$), then they would expect to observe a relatively high price volatility, because errors in private signals would not average out as much as expected, and vice versa.

We show that, if the quadratic variation of actual price dynamics $\hat{P}(t)$ coincides with the quadratic variation of perceived price dynamics $P(t)$, then the incorrect estimates about the number of traders ($N$) cannot be easily falsified by observing the price dynamics.

**Proposition B.1.** *Matching the quadratic variation of actual price dynamics $\hat{P}(t)$ and perceived* *price dynamics $P(t)$ yields the nonrevealing condition,*

$$\frac{1+(\hat{N}-1)\hat{\rho}}{\hat{N}} = \frac{1+(N-1)\rho}{N}. \tag{B-9}$$

*Under this nonrevealing condition, we have*

$$\mathrm{Var}^n[\,\mathrm{d}\hat{P}(t)] = \mathrm{Var}^n[dP(t)], \qquad \mathrm{Cov}^n[\,\mathrm{d}I_n(t),\, \mathrm{d}\hat{P}(t)] = \mathrm{Cov}^n[dI_n(t),\, \mathrm{d}P(t)].$$

If the nonrevealing condition (B-9) is satisfied, then incorrect estimates about the number of traders cannot be easily falsified by observing the price dynamics and its variance. The condition implies that the correlation between trader $n$'s private signal and the actual price change is consistent with the subjective correlation between his private signal and price change. This nonrevealing condition imposes the specific restriction on $\hat{N}, N, \hat{\rho}$, and $\rho$.

If $N < \hat{N}$, then $\rho < \hat{\rho}$, and vice versa. If traders underestimate the total number of participants in the market ($N < \hat{N}$), they should simultaneously underestimate the correlation of the noise terms of their private signals ($\rho < \hat{\rho}$) in order to bring downward the inflated estimate of price change volatility due to the first effect.

As discussed in Section B.2, traders might learn the true total number of traders $N = \hat{N}$ by obtaining some data on residual supply schedules and thus know $\rho = \hat{\rho}$ using non-revealing condition, they are unlikely to know in real time how many of them observe signals with high precision (i.e., $\hat{\theta}$) and thus may easily be mistaken about hard-to-observe parameter $\hat{\theta}$. Therefore, we mainly focus on the case of varying $N_H$ with $N = \hat{N}$ and $\rho = \hat{\rho}$ fixed.

**Proof of Proposition B.1:**

The equilibrium price is determined based on the *actual* market clearing condition, which sums up demands of the actual number of traders $\hat{N}$ in the market,

(B-10) $$\sum_{m=1}^{\hat{N}} x_m(t) = 0, \quad \text{and} \quad \sum_{m=1}^{\hat{N}} S_m(t) = 0.$$

Using equations (A-17), we obtain the actual equilibrium price

(B-11) $$\hat{P}(t) = \frac{\gamma_D(N,\theta,\rho)}{\gamma_P(N,\theta,\rho)} D(t) + \frac{\gamma_H(N,\theta,\rho)}{\gamma_P(N,\theta,\rho)} \breve{a}(N,\theta,\rho) H_0(t) + \frac{\gamma_H(N,\theta,\rho)}{\hat{N}\,\gamma_P(N,\theta,\rho)} \sum_{m=1}^{\hat{N}} H_m(t).$$

Equation (A-19) implies that the actual permanent market depth $1/\hat{\lambda}$ and temporary market depth $1/\hat{\kappa}$, defined as inverse slopes of residual supply functions with respect to the number of shares traded and the rate of trading, are given as

(B-12) $$\frac{1}{\hat{\lambda}} := \frac{(\hat{N}-1)\,\gamma_P}{\gamma_S}, \qquad \frac{1}{\hat{\kappa}} := (\hat{N}-1)\,\gamma_P.$$

By contrast, each trader uses the *subjective* market clearing condition in his calculation summing up demands of the perceived number of traders $N$,

(B-13) $$\sum_{m=1}^{N} x_m(t) = 0, \quad \text{and} \quad \sum_{m=1}^{N} S_m(t) = 0.$$

Each trader believes that the equilibrium price is determined by

(B-14) $$P(t) = \frac{\gamma_D(N,\theta,\rho)}{\gamma_P(N,\theta,\rho)} D(t) + \frac{\gamma_H(N,\theta,\rho)}{\gamma_P(N,\theta,\rho)} \breve{a}(N,\theta,\rho) H_0(t) + \frac{\gamma_H(N,\theta,\rho)}{N\,\gamma_P(N,\theta,\rho)} \sum_{m=1}^{N} H_m(t).$$

The only difference between the two pricing equations (B-11) and (B-14) are indices $\hat{N}$ and $N$ over which the summation of private signals is done.

From equation (A-19), the subjective measures of market depth $1/\lambda$ and $1/\kappa$ are

(B-15)
$$\frac{1}{\lambda} := \frac{(N-1)\,\gamma_P}{\gamma_S}, \qquad \frac{1}{\kappa} := (N-1)\,\gamma_P.$$

Traders believe that markets are deep when the number of traders is large ($N$ is high) and thus they are more willing to provide liquidity to one another ($\gamma_P$ is high).

The only difference between the two pricing equations (B-11) and (B-14) are indices $\hat{N}$ and $N$ over which the summation of private signals is done. For traders not to be able to learn about their own mistakes, the quadratic variation of actual price dynamics $\mathrm{d}\hat{P}(t)$ must coincide with the quadratic variation of perceived price dynamics $\mathrm{d}P(t)$.

Using equations (B-11) and (B-14), we obtain the nonrevealing condition so that the quadratic variation of $\hat{\rho}^{1/2}\,\mathrm{d}Z(t) + (1-\hat{\rho})^{1/2}\frac{1}{\hat{N}}\sum_{m=1}^{\hat{N}}\mathrm{d}B_m(t)$ must coincide with the quadratic variation of $\rho^{1/2}\,\mathrm{d}Z(t) + (1-\rho)^{1/2}\frac{1}{N}\sum_{m=1}^{N}\mathrm{d}B_m(t)$. This implies equation (B-9).

This condition (B-9) also ensures that the correlation coefficient between a trader's private signal and the actual price $\mathrm{Cov}^n[\,\mathrm{d}\hat{I}_n(t),\,\mathrm{d}\hat{P}(t)]$ is consistent with the correlation between his private signal and his "subjective" price $\mathrm{Cov}^n[\,\mathrm{d}I_n(t),\,\mathrm{d}P(t)]$, since

(B-16)
$$\mathrm{Cov}^n[\,\mathrm{d}\hat{I}_n(t),\,\mathrm{d}\hat{P}(t)] = \mathrm{Cov}^n[\hat{\rho}^{1/2}\,\mathrm{d}Z(t) + (1-\hat{\rho})^{1/2}\,\mathrm{d}B_n(t),\ \hat{\rho}^{1/2}\,\mathrm{d}Z(t) + (1-\hat{\rho})^{1/2}\frac{1}{\hat{N}}\sum_{m=1}^{\hat{N}}\mathrm{d}B_m(t)]$$
$$= \hat{\rho} + \frac{1}{\hat{N}}(1-\hat{\rho}).$$

(B-17)

$$\mathrm{Cov}^n[\,\mathrm{d}I_n(t),\,\mathrm{d}P(t)] = \mathrm{Cov}^n[\rho^{1/2}\,\mathrm{d}Z(t) + (1-\rho)^{1/2}\,\mathrm{d}B_n(t),\ \rho^{1/2}\,\mathrm{d}Z(t) + (1-\rho)^{1/2}\tfrac{1}{N}\sum_{m=1}^{N}\mathrm{d}B_m(t)]$$

$$= \rho + \tfrac{1}{N}(1-\rho).$$

This concludes the proof.

## B.4. The Equilibrium with Perfect Competition

Theorem B.2 characterizes the equilibrium of an otherwise equivalent continuous-time model in which the assumption of perfect competition replaces imperfect competition.

**Theorem B.2.** ***Equilibrium of the Continuous-time Model with Perfect Competition.*** *Assume $\tau_H > \tau_L$. There exists a steady-state, Bayesian-perfect equilibrium with symmetric, linear strategies with positive trading volume if and only if the polynomial equation system (B-26) has a solution. Such an equilibrium has the following properties:*

1. *There is an endogenously determined constant $C_L^c > 0$, defined in equation (B-24), such that trader $n$'s optimal inventories $S_n^c(t)$ are*

(B-18)
$$S_n^c(t) = C_L^c\left(\check{H}_n(t) - \check{H}_{-n}(t)\right).$$

2. *There is an endogenously determined constant $C_G^c > 0$, defined in equation (B-23), such that the equilibrium price is*

(B-19)
$$P^c(t) = \frac{D(t)}{r+\alpha_D} + C_G^c\,\frac{\sum_{n=1}^{N} G_n(t)}{N(r+\alpha_D)(r+\alpha_G)}.$$

Price-taking competitors do not smooth their trading. Each trader immediately adjusts actual inventories to target levels (as if $\gamma_S \to \infty$). Equations (B-18) and (B-19) are similar to corresponding equations (29) and (30) in Theorem 2, but the values of $C_L^c$ and $C_G^c$ are different. Competition makes price dampening more pronounced, $C_G^c < C_G$.

Since inventories are diffusions in the continuous-time model, we can think of the ratio of the expected holding position in the equilibrium with imperfect competition to that with perfect competition to measure the competitiveness of the market, $\mathrm{E}|S_n(t)|/\mathrm{E}|S_n^c(t)|$, where $S_n(t)$ is given in Theorem 2 and $S_n^c(t)$ is given in Theorem B.2.

**Proof of Theorem B.2:**

We conjecture that price is a linear function of $D(t)$ and $\frac{1}{N}\sum_{n=1}^{N} G_n(t)$,

(B-20) $$P(t) = \frac{D(t)}{r+\alpha_D} + C_G^c \, \frac{\sum_{n=1}^{N} G_n(t)}{N(r+\alpha_D)(r+\alpha_G)}.$$

We conjecture and verify that the value function $V(W_n, \check{H}_n, \check{H}_{-n})$ has the specific quadratic exponential form

(B-21) $$V(W_n, \check{H}_n, \check{H}_{-n}) = -\exp\Big(\psi_0 + \psi_W W_n + \tfrac{1}{2}\psi_{nn}(\check{H}_n)^2 + \tfrac{1}{2}\psi_{xx}(\check{H}_{-n})^2 + \psi_{nx}\check{H}_n\check{H}_{-n}\Big).$$

The five constants $\psi_0$, $\psi_W$, $\psi_{nn}$, $\psi_{xx}$, and $\psi_{nx}$ have values consistent with a steady-state equilibrium. The terms $\psi_{nn}$, $\psi_{xx}$, and $\psi_{nx}$ capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term $\psi_0$. The Hamilton–Jacobi–Bellman (HJB) equation corresponding to the conjectured value function $V(W_n, \check{H}_n, \check{H}_{-n})$ in equation (B-21) is

$$
\begin{aligned}
0 = \min_{c_n, s_n} \; & -\frac{e^{-Ac_n}}{V} - \rho + \psi_W \Big( rW_n + S_n D(t) - c_n - rP(t)S_n(t) - \tfrac{\alpha_D}{r+\alpha_D} D(t) S_n \\
& + \tfrac{\sigma_G \Omega^{1/2}}{r+\alpha_D} \Big( \check{\tau}_H^{1/2} \check{H}_n(t) + (N-1)\check{\tau}_L^{1/2} \check{H}_{-n}(t) \Big) S_n \\
& + \tfrac{C_G^c \sigma_G \Omega^{1/2} (\check{\tau}_H^{1/2} + (N-1)\check{\tau}_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \big( (a_1 + (N-1)a_4)\check{H}_n(t) + (a_3 + (N-1)a_2)\check{H}_{-n}(t) \big) S_n \Big) \\
& + \big( \psi_{nn} \check{H}_n(t) + \psi_{nx} \check{H}_{-n}(t) \big) \Big( -(\alpha_G + \tau)\check{H}_n + (\tau_H^{1/2} + \check{a}\tau_0^{1/2})(\check{\tau}_H^{1/2} \check{H}_n + (N-1)\check{\tau}_L^{1/2} \check{H}_{-n}) \Big) \\
& + \big( \psi_{xx} \check{H}_{-n}(t) + \psi_{nx} \check{H}_n(t) \big) \Big( -(\alpha_G + \tau)\check{H}_{-n} + (\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2} + \check{a}\tau_0^{1/2})(\check{\tau}_H^{1/2} \check{H}_n + (N-1)\check{\tau}_L^{1/2} \check{H}_{-n}) \Big) \\
& + \tfrac{1}{2}\psi_W^2 S_n^2 \Big( \tfrac{(C_G^c)^2 \sigma_G^2 \Omega (N\check{a}^2 + (N-1)\rho + 1)(\check{\tau}_H^{1/2} + (N-1)\check{\tau}_L^{1/2})^2}{N(r+\alpha_D)^2 (r+\alpha_G)^2} + \tfrac{\sigma_D^2}{(r+\alpha_D)^2} + \tfrac{2C_G^c \sigma_G \sigma_D \Omega^{1/2} \tau_0^{1/2}}{(r+\alpha_D)^2 (r+\alpha_G)} \Big) \\
& + \tfrac{1}{2} \big( (\psi_{nn} \check{H}_n(t) + \psi_{nx} \check{H}_{-n}(t))^2 + \psi_{nn} \big) \big( 1 + \check{a}^2 \big) + \tfrac{1}{2} \big( (\psi_{xx} \check{H}_{-n}(t) + \psi_{nx} \check{H}_n(t))^2 + \psi_{xx} \big) \Big( \tfrac{1-\rho}{N-1} + \rho + \check{a}^2 \Big) \\
& + \psi_W S_n \big( (\psi_{nn} + \psi_{nx}) \check{H}_n(t) + (\psi_{xx} + \psi_{nx}) \check{H}_{-n}(t) \big) \\
& \qquad \cdot \Big( \tfrac{C_G^c \sigma_G \Omega^{1/2}}{N(r+\alpha_D)(r+\alpha_G)} (\check{\tau}_H^{1/2} + (N-1)\check{\tau}_L^{1/2})(N\check{a}^2 + 1 + (N-1)\rho) + \tfrac{\sigma_D \check{a}}{r+\alpha_D} \Big) \\
& + \big( (\psi_{nn} \check{H}_n(t) + \psi_{nx} \check{H}_{-n}(t)) \, (\psi_{xx} \check{H}_{-n}(t) + \psi_{nx} \check{H}_n(t)) + \psi_{nx} \big) \, (\check{a}^2 + \rho),
\end{aligned}
$$

where constants $a_1$, $a_2$, $a_3$, and $a_4$ are defined as in equation (A-30). The solution for optimal consumption is

$$
c_n(t) = -\tfrac{1}{A} \, \log \Big( \tfrac{\psi_W \, V(t)}{A} \Big).
$$

Plugging optimal consumption and $P(t)$ from equation (B-20) into the HJB equation yields a quadratic function of $S_n$. It can be shown that the optimal trading strategy is a linear function of the state variables $\breve{H}_n(t)$ and $\breve{H}_{-n}(t)$,

(B-22)
$$\begin{aligned} S_n(t) = C\Big( & C_G^c \sigma_G \Omega^{1/2} \big(\breve{\tau}_H^{1/2} + (N-1)\breve{\tau}_L^{1/2}\big) \\ & \cdot \Big(\big(r - a_1 - (N-1)a_4\big)\breve{H}_n(t) + \big((N-1)(r-a_2) - a_3\big)\breve{H}_{-n}(t)\Big) \\ & - \sigma_G \Omega^{1/2} (r + \alpha_G) N \Big(\breve{\tau}_H^{1/2} \breve{H}_n(t) + (N-1)\breve{\tau}_L^{1/2} \breve{H}_{-n}(t)\Big) \\ & - \Big((\psi_{nn} + \psi_{nx})\breve{H}_n(t) + (\psi_{xx} + \psi_{nx})\breve{H}_{-n}(t)\Big) \\ & \cdot \Big(C_G^c \sigma_G \Omega^{1/2} \big(\breve{\tau}_H^{1/2} + (N-1)\breve{\tau}_L^{1/2}\big)(N\breve{a}^2 + 1 + (N-1)\rho) + \sigma_D \breve{a} N (r + \alpha_G)\Big)\Big), \end{aligned}$$

where $\breve{\tau}_H^{1/2}$ and $\breve{\tau}_L^{1/2}$ are defined in equation (A-16) and

$$C = \frac{(r+\alpha_D)(r+\alpha_G)/\psi_W}{(C_G^2)^2 \sigma_G^2 \Omega \Big(\breve{\tau}_H^{1/2} + (N-1)\breve{\tau}_L^{1/2}\Big)^2 (N\breve{a}^2 + 1 + (N-1)\rho) + N\sigma_D^2 (r+\alpha_G)^2 + 2N(r+\alpha_G)\sigma_D C_G^c \sigma_G \Omega^{1/2} \tau_0^{1/2}}.$$

Since the market clears, $\sum_{n=1}^N S_n(t) = 0$, this implies

(B-23)
$$C_G^c = \frac{N(r+\alpha_G)\Big(\sigma_G \Omega^{1/2} + \sigma_D \breve{a}(\psi_{nn} + \psi_{xx} + 2\psi_{nx}) / \Big(\hat{\tau}_H^{1/2} + (N-1)\hat{\tau}_L^{1/2}\Big)\Big)}{\sigma_G \Omega^{1/2} (Nr - a_1 - a_3 - (N-1)(a_2 + a_4) - (1 + (N-1)\rho + N\breve{a}^2)(\psi_{nn} + \psi_{xx} + 2\psi_{nx}))}.$$

Combining equations (B-22) and (B-23), we get $S_n(t) = C_L^c \big(\breve{H}_n(t) - \breve{H}_{-n}(t)\big)$, where the constant $C_L^c$ is defined as

(B-24)
$$\begin{aligned} C_L^c = C\Big( & \sigma_G \Omega^{1/2} \Big(C_G^c \big(\breve{\tau}_H^{1/2} + (N-1)\breve{\tau}_L^{1/2}\big)\big(r - a_1 - (N-1)a_4\big) - N\breve{\tau}_H^{1/2}(r + \alpha_G)\Big) \\ & - (\psi_{nn} + \psi_{nx}) \Big(C_G^c \sigma_G \Omega^{1/2} \big(\breve{\tau}_H^{1/2} + (N-1)\breve{\tau}_L^{1/2}\big)(1 + (N-1)\rho + N\breve{a}^2) + \sigma_D \breve{a} N (r + \alpha_G)\Big)\Big). \end{aligned}$$

Plugging (B-22) back into the Bellman equation and setting the constant term and the coefficients of $W_n$, $(\breve{H}_n)^2$, $(\breve{H}_{-n})^2$, and $\breve{H}_n\, \breve{H}_{-n}$ to be zero, we obtain five equations, from which we can find five unknown parameters $\psi_0, \psi_W, \psi_{nn}, \psi_{nx}$ and $\psi_{xx}$. By setting the constant term and coefficient of $W_n$ to be zero, we obtain

(B-25)

$$\psi_W = -rA, \qquad \psi_0 = 1 - \log(r) + \frac{1}{r}\Big(-\rho + \tfrac{1}{2}(1+\breve{a}^2)\psi_{nn} + \tfrac{1}{2}\Big(\rho + \tfrac{1-\rho}{N-1} + \breve{a}^2\Big)\psi_{xx} + (\breve{a}^2+\rho)\psi_{nx}\Big).$$

By setting the coefficients of $(\breve{H}_n)^2, (\breve{H}_{-n})^2$ and $\breve{H}_n\, \breve{H}_{-n}$ to be zero, we obtain three polynomial equations in the three unknowns $\psi_{nn}, \psi_{xx}$, and $\psi_{nx}$. Defining $c_1$, $c_2$, $c_3$, and $c_4$ by

$$c_1 = \tfrac{(C_G^c)^2\sigma_G^2\Omega(N\breve{a}^2+1+(N-1)\rho)(\breve{\tau}_H^{1/2}+(N-1)\breve{\tau}_L^{1/2})^2}{N(r+\alpha_D)^2(r+\alpha_G)^2} + \tfrac{\sigma_D^2}{(r+\alpha_D)^2} + \tfrac{2C_G^c\sigma_G\sigma_D\Omega^{1/2}\tau_0^{1/2}}{(r+\alpha_D)^2(r+\alpha_G)},$$

$$c_2 = \tfrac{C_G^c\sigma_G\Omega^{1/2}}{N(r+\alpha_D)(r+\alpha_G)}(\breve{\tau}_H^{1/2} + (N-1)\breve{\tau}_L^{1/2})(N\breve{a}^2 + 1 + (N-1)\rho) + \tfrac{\sigma_D\breve{a}}{r+\alpha_D},$$

$$c_3 = \tfrac{rA\sigma_G\Omega^{1/2}C_L^c}{r+\alpha_D}\Big(\tfrac{C_G^c(\breve{\tau}_H^{1/2}+(N-1)\breve{\tau}_L^{1/2})(r-a_1-(N-1)a_4)}{N(r+\alpha_G)} - \breve{\tau}_H^{1/2}\Big),$$

$$c_4 = \tfrac{rA\sigma_G\Omega^{1/2}C_L^c}{r+\alpha_D}\Big(\tfrac{C_G^c(\breve{\tau}_H^{1/2}+(N-1)\breve{\tau}_L^{1/2})(r-a_2-\frac{a_3}{N-1})}{N(r+\alpha_G)} - \breve{\tau}_L^{1/2}\Big),$$

these three equations in three unknowns can be written as follows:

(B-26)

$$0 = -\tfrac{r}{2}\psi_{nn} + a_1\psi_{nn} + a_4\psi_{nx} - rAC_L^c c_2(\psi_{nn}+\psi_{nx}) + \tfrac{1}{2}(1+\breve{a}^2)\psi_{nn}^2$$
$$+ \tfrac{1}{2}\left(\rho + \tfrac{1-\rho}{N-1} + \breve{a}^2\right)\psi_{nx}^2 + (\breve{a}^2+\rho)\psi_{nn}\psi_{nx} + c_3 + \tfrac{1}{2}r^2A^2c_1(C_L^c)^2,$$

$$0 = -\tfrac{r}{2}\psi_{xx} + a_2\psi_{xx} + a_3\psi_{nx} + rAC_L^c c_2(\psi_{xx}+\psi_{nx}) + \tfrac{1}{2}(1+\breve{a}^2)\psi_{nx}^2$$
$$+ \tfrac{1}{2}\left(\rho + \tfrac{1-\rho}{N-1} + \breve{a}^2\right)\psi_{xx}^2 + (\breve{a}^2+\rho)\psi_{xx}\psi_{nx} - (N-1)c_4 + \tfrac{1}{2}r^2A^2c_1(C_L^c)^2,$$

$$0 = -r\psi_{nx} + (a_1+a_2)\psi_{nx} + a_3\psi_{nn} + a_4\psi_{xx} + rAC_L^c c_2(\psi_{nn}-\psi_{xx})$$
$$+ (1+\breve{a}^2)\psi_{nn}\psi_{nx} + \left(\rho + \tfrac{1-\rho}{N-1} + \breve{a}^2\right)\psi_{xx}\psi_{nx} + (\breve{a}^2+\rho)\left(\psi_{nn}\psi_{xx} + \psi_{nx}^2\right)$$
$$+ (N-1)c_4 - c_3 - r^2A^2c_1(C_L^c)^2.$$

Parameters $\psi_{nn}, \psi_{nx}, \psi_{xx}$ are solved numerically from the system of equation (B-26).

## B.5. Price Dampening and Return Autocorrelations

Using equation (30), we derive the holding period return $R(t, t+T)$ from $t$ to $t+T$ and show that the return autocorrelations are positive.

In the dynamic process of sufficient statistics $\mathrm{d}H_n(t)$ and $\mathrm{d}H_{-n}(t)$ (equation (A-14)), the coefficient $\tau_H^{1/2}$ in the second term on the right hand side of $\mathrm{d}H_n(t)$ is different from the coefficient $\theta\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}$ in the second term on the right hand side of $\mathrm{d}H_{-n}(t)$. This difference is the key driving force behind the price-dampening effect resulting from the Keynesian beauty contest. It captures the fact that—in addition to disagreeing about the value of the asset in the present—traders also disagree about the dynamics of their future valuations. We use traders' beliefs to study return dynamics.

We assume that $N = \hat{N}$, $\rho = \hat{\rho}$, but $\theta \neq \hat{\theta}$, *i.e.,* traders might misestimate the fraction of traders who are trading in the same strategy space. From equations (22) and (A-13), trader $n$ believes that the dynamics of $\bar{G}(t) := \frac{1}{N}\sum_{n=1}^{N} G_n(t)$, $G_n(t)$, and $G_{-n}(t) := \frac{1}{N-1}\sum_{m=1, m\neq n}^{N} G_m(t)$ are given by

$$
\begin{aligned}
\mathrm{d}\bar{G}(t) =& -(\alpha_G + \tau)\bar{G}(t)\,\mathrm{d}t + \big(\tau_0 + N(1+(N-1)\rho)\check{\tau}_I\big)G_n(t)\,\mathrm{d}t \\
&+ \sigma_G \Omega^{1/2}\left(\check{\tau}_I^{1/2}\,\mathrm{d}B_n^n(t) + \tau_0^{1/2}\,\mathrm{d}B_0^n(t) + \check{\tau}_I^{1/2}\sum_{m=1, m\neq n}^{N}\mathrm{d}B_m^n(t)\right), \\
\mathrm{d}G_n(t) =& -\alpha_G G_n(t)\,\mathrm{d}t + \sigma_G\Omega^{1/2}\left(\check{\tau}_H^{1/2}\,\mathrm{d}B_n^n(t) + \tau_0^{1/2}\,\mathrm{d}B_0^n(t) + \check{\tau}_L^{1/2}\sum_{m=1, m\neq n}^{N}\mathrm{d}B_m^n(t)\right), \\
\mathrm{d}G_{-n}(t) =& -\left(\alpha_G + \tfrac{(1-\theta)^2}{1-\rho}\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right)G_{-n}(t)\,\mathrm{d}t + \left(\tau - \tfrac{(1-\theta)^2}{1-\rho}\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right)\big(G_n(t) - G_{-n}(t)\big)\,\mathrm{d}t \\
&+ \sigma_G\Omega^{1/2}\left(\tau_0^{1/2}\,\mathrm{d}B_0^n(t) + \check{\tau}_L^{1/2}\,\mathrm{d}B_n^n(t) + \tfrac{\check{\tau}_H^{1/2} + (N-2)\check{\tau}_L^{1/2}}{N-1}\sum_{m=1, m\neq n}^{N}\mathrm{d}B_m^n(t)\right), \quad \text{where}
\end{aligned}
$$

$$
\check{\tau}_H^{1/2} := \tfrac{(1-\theta)(\tau_H^{1/2} - \tau_L^{1/2})}{1-\rho} + \tfrac{(\theta-\rho)\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}, \quad \check{\tau}_L^{1/2} := \tfrac{(\theta-\rho)\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}, \qquad \check{\tau}_I^{1/2} := \tfrac{1}{N}\check{\tau}_H^{1/2} + \tfrac{N-1}{N}\check{\tau}_L^{1/2}.
$$

Using equation (30), it can be shown that trader $n$ believes that the instantaneous returns process is a linear combination of $G_n(t)$ and $\bar{G}(t)$:

(B-27)
$$\begin{aligned} &\mathrm{d}P(t)+D(t)\,\mathrm{d}t-rP(t)\,\mathrm{d}t \\ &=\Big(\tfrac{1}{r+\alpha_D}+\tfrac{C_G(\tau_0+N(1+(N-1)\rho)\check{\tau}_I)}{(r+\alpha_D)(r+\alpha_G)}\Big)G_n(t)\,\mathrm{d}t-C_G\Big(\tfrac{1}{r+\alpha_D}+\tfrac{\tau}{(r+\alpha_D)(r+\alpha_G)}\Big)\bar{G}(t)\,\mathrm{d}t+\mathrm{d}B_r(t), \end{aligned}$$

where $\mathrm{d}B_r(t)$ is defined as follows:

(B-28)
$$\mathrm{d}B_r(t):=\tfrac{\sigma_D}{r+\alpha_D}\,\mathrm{d}B_0^n(t)+\tfrac{\sigma_G C_G\Omega^{1/2}}{(r+\alpha_D)(r+\alpha_G)}\left(\tau_0^{1/2}\,\mathrm{d}B_0^n(t)+\check{\tau}_I^{1/2}\Big(\mathrm{d}B_n^n(t)+\sum_{m=1,m\neq n}^{N}\mathrm{d}B_m^n(t)\Big)\right).$$

Integrating equation (B-27) over time yields the holding period return from $t$ to $t+T$:

(B-29)
$$R(t,t+T)=\int_{u=t}^{t+T}\mathrm{e}^{r(t+T-u)}\big(\mathrm{d}P(u)+D(u)\,\mathrm{d}u-rP(u)\,\mathrm{d}u\big).$$

Define a continuous 2-vector stochastic process $y_G(t)=[G_n(t)\,,\,\bar{G}(t)]'$. It can be shown that $y_G(t)$ satisfies the linear stochastic differential equation

$$\mathrm{d}y_G(t)=K_G y_G(t)\,\mathrm{d}t+C_Z\,\mathrm{d}Z_G(t),$$

where $K_G$ is a $2\times 2$ matrix and $C_Z$ is a $2\times 3$ matrix given by

$$K_G=\begin{bmatrix} -\alpha_G & 0 \\ \tau_0+N(1+(N-1)\rho)\check{\tau}_I & -(\alpha_G+\tau) \end{bmatrix},\qquad C_Z=\sigma_G\Omega^{1/2}\begin{bmatrix} \tau_0^{1/2} & \check{\tau}_H^{1/2} & (N-1)^{1/2}\check{\tau}_L^{1/2} \\ \tau_0^{1/2} & \check{\tau}_I^{1/2} & (N-1)^{1/2}\check{\tau}_I^{1/2} \end{bmatrix}.$$

The vector $\mathrm{d}Z_G(t)=[\,\mathrm{d}B_0^n(t)\,,\ \mathrm{d}B_n^n(t)\,,\,(N-1)^{-1/2}\sum_{m=1,m\neq n}^{N}\mathrm{d}B_m^n(t)]'$ is a $3\times 1$-dimensional

Brownian motion. We can then obtain recursive formulas for the stochastic vector $y_G(u) = [G_n(u)\,,\, \bar{G}(u)]'$ for time $u \geq t$ as a function of $y_G(t) = [G_n(t)\,,\, \bar{G}(t)]'$.

Plugging these recursive formulas into equation (B-29), we obtain the cumulative holding period return $R(t, t+T)$ over period $T$ between time $t$ and $t+T$ as a linear function of $G_n(t)$ and $\bar{G}(t)$:

(B-30)
$$R(t, t+T) = \zeta_2(T)G_n(t) - \zeta_1(T)\bar{G}(t) + \bar{B}(t, t+T).$$

The coefficients $\zeta_1(T) > 0$ and $\zeta_2(T) > 0$ in the equation above are defined as

(B-31)
$$\zeta_1(T) := \frac{C_G}{(r+\alpha_G)(r+\alpha_D)}\left(\mathrm{e}^{rT} - \mathrm{e}^{-(\alpha_G+\tau)T}\right),$$
$$\zeta_2(T) := \frac{\mathrm{e}^{-(\alpha_G+\tau)T}}{(r+\alpha_G)(r+\alpha_D)\tau}\left(\left(\mathrm{e}^{(r+\alpha_G+\tau)T} - \mathrm{e}^{\tau T}\right)\tau + C_G\left(\tau_0 + N(1+(N-1)\rho)\check{\tau}_I\right)\left(\mathrm{e}^{\tau T} - 1\right)\right),$$

and the Brownian motion term $\bar{B}(t, t+T)$ is defined as

(B-32)
$$\bar{B}(t, t+T) := \int_{s=t}^{t+T}\int_{u=s}^{t+T}[a, -b]\,\mathrm{e}^{r(t+T-u)+K_G(u-s)}\,C_Z\,\mathrm{d}u\,\mathrm{d}Z_G(s) + \int_{s=t}^{t+T}\mathrm{e}^{r(t+T-s)}\,\mathrm{d}B_r(s),$$

where constants $a$ and $b$ are defined as follows:

$$a := \frac{1}{r+\alpha_D} + \frac{C_G(\tau_0 + N(1+(N-1)\rho)\check{\tau}_I)}{(r+\alpha_D)(r+\alpha_G)}, \quad b := C_G\left(\frac{1}{r+\alpha_D} + \frac{\tau}{(r+\alpha_D)(r+\alpha_G)}\right).$$

We then derive the autocorrelations of the cumulative returns, $\mathrm{Corr}^n[R(t-T, t), R(t, t+T)]$. We numerically show that the return autocorrelations are positive for a wide range of parameter values, implying that the return exhibits time-series return momentum.