Index Investing and Asset Pricing under Information Asymmetry and Ambiguity Aversion

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Abstract

In a setting with a tradable value-weighted market index, ambiguity averse investors do not trade, and the index is not mean-variance efficient. But when a passive fund offers the risk-adjusted market portfolio (*RAMP*), whose weights depend on information precisions as well as market values, investors share risk via index investing and effectively hold the same portfolios as in the economy without model uncertainty. This follows from a new *Information Separation Theorem*: equilibrium portfolios are the sum of *RAMP*, which is the optimal portfolio conditional on public information, and their optimal private-information-based portfolios. *RAMP* is in equilibrium mean-variance efficient.

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I. Introduction

Index investing has long been recommended by distinguished practitioners, such as Jack Bogle and Warren Buffett, and leading scholars, such as William Sharpe and Harry Markowitz, as a means for retail investors to optimize risk and return. In recent years, investors have increasingly followed this recommendation (Jiang, Vayanos, and Zheng, 2020). Market indexes are also central to asset pricing in determining asset risk premia.

These two roles of market indexes, facilitating stock market investment and pricing assets in the cross section, flow naturally from existing theory. For example, in the Capital Asset Pricing Model (Sharpe, 1964; Lintner, 1965), in equilibrium, all investors hold the *Value-Weighted Market Portfolio* (*VWMP*), and with the *VWMP* as the pricing portfolio, correctly priced assets have zero alphas.

However, when the CAPM assumptions are violated, the *VWMP* is dysfunctional. In practice, investors have heterogeneous information, and their private signals can differ in precision. So even if informed investors were rational, if there is some noise in the system, investors would not hold the same portfolio. Indeed, Admati (1985) shows that in such a setting, the *VWMP* is not mean-variance efficient for any information set.

Furthermore, laboratory and field evidence indicates that investors usually face model uncertainty (also known as ambiguity; i.e., investors do not know exactly financial market parameters). When investors are ambiguous about certain assets, they may perceive such assets as very risky and thus take zero positions in these assets either directly or indirectly, even if the *VWMP* is offered to them. The conclusion that ambiguity aversion reduces market participation is suggested by experimental evidence, such as Dimmock, Kouwenberg, and Wakker (2016), Bianchi and Tallon

(2019), and Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019). Empirical studies such as Dimmock, Kouwenberg, Mitchell, and Peijnenburg (2016) also show that ambiguity aversion is negatively associated with stock market participation and is associated with portfolio underdiversification. This suggests that when investors are ambiguity averse, the *VWMP* may function poorly as a mechanism for promoting participation and risk-sharing.

In this paper, we study index investing and asset pricing under asymmetric information and ambiguity aversion. We address the following questions. First, in the financial market where investors commonly know financial market parameters but hold heterogeneous information about asset returns, is there a different index that can replace the dysfunctional *VWMP*, facilitating index investing and serving as a pricing portfolio for the CAPM pricing relation? If so, how is it constructed? Second, when investors are not only asymmetrically informed about asset returns but also ambiguity averse about financial market parameters, can we usefully decompose their equilibrium portfolios into components held to share risk versus to exploit private information? Third, is there a constructed index that all investors will optimally hold as the common component of their portfolios which has the property that assets are priced relative to it?

We address these questions in a noisy rational expectations equilibrium setting with investors who do not know some parameters of the financial market, and are ambiguity averse, meaning that investors optimize under worst-case assumptions about these parameters (Gilboa and Schmeidler, 1989). We introduce a new index portfolio, whose weights are functions of the financial market parameters. Consistent with institutional facts, we assume that the index is constructed and maintained by a professional index committee and is replicated and offered to investors by a passive index fund. We assume that the index committee specializes in understanding financial markets, and acquires knowledge about all assets, which is sufficient to construct the index.

We start with a benchmark setting in which investors all know the financial market parameters. This is essentially a traditional rational expectations equilibrium model. As in the literature, in equilibrium, the *VWMP* is not generically mean-variance efficient regardless of the information set. This detracts severely from it a usefulness, when offered for trading as an index, as a vehicle for diversification and risk-sharing across heterogeneous investors. Furthermore, conditional on public information, with the *VWMP* being the pricing portfolio, risky assets have non-zero alphas in the CAPM pricing relation.

We offer a new separation theorem, applicable in this setting (and generalizable to allow for model uncertainty), which guides the construction of a new index that facilitates index investing and serves as a pricing portfolio. In equilibrium, an investor's portfolio holding is the sum of two portfolios. The first is optimal conditional only on the public information signal.[1] The second is optimal conditional only on her private information, not on public information as reflected in market price. We call the former the *Risk Adjusted Market Portfolio* (or *RAMP*) and the latter the *information-based portfolio*. Since *RAMP* is independent of any investor's private signal, it can be constructed and offered by a passive index fund. Our separation theorem differs from separation results derived in the literature in that an investor's optimal portfolio is separated into two components that are optimal portfolios conditional upon the public information, and conditional upon the investor's private information, respectively. We therefore call it the *Information Separation Theorem*.

Using this theorem, we derive *RAMP* as the index portfolio that is optimal for investors to hold based solely on public information. In *RAMP*, asset weights depend upon the average precision of

[1]For simplicity, our model assumes a diffuse prior and no exogenous public signals, making the price signal the sole source of public information. However, as shown in the appendix, our results remain valid even when additional exogenous public signals are introduced.

investor private signals and the precisions of random supply shocks, as well as the market values of assets. Relative to the *VWMP*, *RAMP* has lower investment in more volatile assets (conditional on asset prices), so it is a defensive (low volatility) investing strategy. As such, *RAMP* can be viewed as a "smart beta" investment strategy, a general approach that has gained great popularity in practice.[2]

RAMP facilitates index investing to diversify and share risk. Intuitively, since *RAMP* is the optimal portfolio conditional only on public information, an index fund offering it will attract investors who do not possess private signals. Furthermore, by the Information Separation Theorem, even investors with private information can construct their overall portfolios by directly forming their personal information-based portfolios, together with holding one share of a *RAMP* index fund.

Furthermore, we show that the CAPM pricing relation holds with *RAMP* as the pricing portfolio, even though there is asymmetric information across investors. This result follows from the fact that *RAMP* is mean-variance efficient conditional on the public information.

Our *RAMP* version of the CAPM differs markedly from other versions of the CAPM that have been derived in related model setups (see, for example, Easley and O'hara (2004), Biais, Bossaerts, and Spatt (2010), and the online appendix of Van Nieuwerburgh and Veldkamp (2010)). In these models, the market portfolio for CAPM pricing is the ex-post total supply of the risky assets, the sum of the endowed risky assets and the random supply of risky assets. This market portfolio differs from *RAMP*, which is not the aggregate total supply of risky assets. These authors show that

[2]Unlike traditional index funds, which weight securities based on their market capitalizations, smart beta funds aim to provide better risk-adjusted returns by incorporating characteristics such as firm size, value proxies, and volatility into security weights. According to Morningstar, by the end of 2021 smart beta funds surpassed $1.45 trillion in assets under management.

this market portfolio is mean-variance efficient conditional on the average investor's information set, and so the CAPM security market line holds from the perspective of the average investor. However, neither the average investor's information nor the pricing portfolio is observable, so such a CAPM pricing relation cannot be directly tested (Biais et al., 2010). Andrei, Cujean, and Wilson (2023) derive another version of CAPM pricing relation in a similar framework, where the pricing portfolio is the *VWMP*, but the asset betas are defined differently from the traditional CAPM.

Our version of CAPM has a different pricing portfolio, *RAMP*, and defines beta exactly as in the traditional CAPM (the ratio of the covariance between the individual asset return and the pricing portfolio to the variance of the pricing portfolio). As *RAMP* does not depend on the realization of the random supply shock, and the weight of each asset in the pricing portfolio is a function of public observables rather than on supply shocks, the portfolio is potentially observable to an econometrician. This makes the model empirically testable.

In our main model, investors do not know some or all of the financial market parameters. For example, we allow for investor ignorance of the average precision of private signals about asset returns among all investors, or of the precisions of the random supply shocks. Furthermore, if an investor is ambiguous about an asset, she does not possess a private signal about its return. However, for each asset, there is a positive measure of investors who know the financial market parameters and possess private signals about the asset's return. Additionally, we allow for an index fund whose management knows the financial market parameters but has no private signals about asset returns. The fund commits to offering either the *VWMP* or the *RAMP*.

When the fund offers the *VWMP*, in equilibrium, investors take zero positions in the assets that they are ambiguous about, and also take a zero position in the fund. In equilibrium investors still hold the aggregate supply of each asset, since there are investors who are not ambiguous about each asset. As in rational expectations equilibrium models such as Grossman and Stiglitz (1980), Hellwig (1980), and Admati (1985), to assess the risk level of *VWMP*, investors need to know all parameters that determine the risky asset pricing function. However, when investors face model uncertainty and thus do not possess full information about these parameters, they perceive that the *VWMP* to be extremely risky, and evaluate options in terms of worst-case scenarios for ambiguous parameters. So even when a *VWMP* index portfolio is available, ambiguity averse investors do not trade stocks. In addition, since the *VWMP* is not mean-variance efficient with respect to the public information set, assets have non-zero alphas relative to the *VWMP*. These alphas include ambiguity premia that are induced by limited investor participation in the stock market.

We then focus on the case where the index fund offers *RAMP*. It is not obvious whether such a fund would attract ambiguity averse investors. Just as with the *VWMP*, *RAMP* may potentially be perceived to be extremely risky. Furthermore, investors differ in their beliefs about the structure of the financial market and have different private signals about asset payoffs. So it is unclear whether one passive fund can be useful for all investors. In addition, investor ambiguity aversion in our model is a departure from the traditional CARA-normal setting, so tractability is not obvious.

Nevertheless, we show that there exists an equilibrium in which investors' asset holdings are exactly the same as those in the economy without model uncertainty. All investors, whether facing model uncertainty or not, delegate the passive component of their portfolios to the index fund by holding exactly one share of *RAMP*. Investors additionally hold their information-based portfolios. So partial delegation by investors of portfolio choice to the index fund solves the nonparticipation problem caused by ambiguity aversion.

A surprising property of the equilibrium is that investors hold the same number of shares of the fund. In equilibrium, investors may disagree intensely for two reasons. First, investors have different private signals about asset payoffs. Second, a pair of investors who have non-overlapping subjective belief supports about financial market parameters will disagree about the portfolio weights in *RAMP* (which are functions of these parameters). So investors form different estimates of the risk and return of *RAMP*. Despite this heterogeneity, investors in equilibrium choose to invest identically in the *RAMP* index fund.

Furthermore, we show all assets have zero alphas relative to *RAMP*. It follows from the Information Separation Theorem that an investor with no private information optimally holds *RAMP* (through the fund). This implies that *RAMP* is mean-variance efficient for all possible values of the ambiguous parameters. So the CAPM security market line holds with *RAMP* as the pricing portfolio regardless of the actual values of the financial market parameters.

The key intuition for why investors hold *RAMP* as the passive component of their portfolios derives from the Information Separation Theorem. Consider the strategy profile in which each investor holds exactly one share of the fund, and additionally holds her information-based portfolio (which could be a nullity). Given that all other investors behave as prescribed, no investor has an incentive to deviate.

In this setting, the fund provides investors with a channel to share risks. Consider, for example, an investor and a vector of parameters of a financial market that is possible according to her subjective belief. Given the value of this vector, the investor would be in a possible world without model uncertainty. Since all other investors are holding one share of the fund and their own information-based portfolios, they are effectively holding the same portfolios as they would in the rational expectations equilibrium in this world, even if they perceive different values of the vector. Hence, the market clearing condition implies that the pricing function is the same as the one in the rational expectations equilibrium. Therefore, if the investor knew the parameter values that characterize this possible world, her optimal portfolio choice would consist of *RAMP* and her own information-based portfolio.

Because the fund offers *RAMP* in any possible world, without the knowledge of the exact world, the investor's optimal portfolio choice is to hold one share of the fund together with her own information-based portfolio. Investors may disagree with each other about financial market parameters (owing to having different prior supports of their distributions) and thus the fund's composition and risks. Nevertheless, by holding one share of the fund along with their own information-based portfolios, each investor knows that her position is, in the proposed equilibrium, optimal under each possible value of the financial market parameter vector. When other investors trade as prescribed, the optimal non-informational component of the investment strategy, assuming that the investor knows the financial market parameters, is independent of the particular values of these parameters. This implies that this portfolio component maximizes the investor's utility in the worst-case scenario, or more technically, solves the investor's max-min utility. Put differently, a strong max-min property holds in the equilibrium. Therefore, ambiguity averse investors all hold the same diversified portfolio, *RAMP*, by holding the passive fund.

The above argument makes clear that an investor's willingness to hold the fund relies crucially on equilibrium reasoning. Each investor understands that all other investors hold the fund and their own information-based portfolios. Indeed, if an ambiguity averse investor thought that other investors were not investing as prescribed in the equilibrium, she would not in general find it optimal to hold the fund.

Our findings contribute to the growing literature on the economic consequences of index investment (Chabakauri and Rytchkov (2021), Bond and Garcia (2022), Baruch and Zhang (2022)).

These papers assume that some investors exogenously can trade only the value-weighted index portfolio, not individual stocks; or that index investing is traded because doing so is assumed to be less costly. In contrast, in our model that all investors can freely decide whether to employ an index investing strategy, and analyzes how a newly designed index, *RAMP*, affects portfolio choice and asset pricing in financial markets with information asymmetry and ambiguity averse investors.

Past literature has studied extensively whether ambiguity aversion hinders market participation and causes socially inefficient risk sharing. Early work on market participation mainly considers investor uncertainty about the first moment of asset payoffs in partial equilibrium frameworks.[3] Recent papers, such as Condie and Ganguli (2011), Condie and Ganguli (2017), and Illeditsch, Ganguli, and Condie (2021), study the effects of investor ambiguity about the characteristics of private or public information in rational expectations equilibrium settings, and show that asset prices may be informationally inefficient. Our paper differs in considering a more general set of parameters for investor model uncertainty—those that characterize asset payoff distributions, and in studying index investing as a way to address ambiguity aversion.

Our setting endogenizes a type of investor trust in asset markets (Gennaioli, Shleifer, and Vishny, 2015). It is well understood that to induce investors to hold an equity fund, investors must trust in the honesty and competence of the fund managers. A further insight of our model is that it is also crucial that investors foresee an equilibrium in which other investors also trust the fund managers and trade accordingly. Off equilibrium, an investor would not be willing to hold the fund, even if she trusted the fund manager.

[3] See, e.g., Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010), Cao, Wang, and Zhang (2005), Dow and da Costa Werlang (1992), Easley and O'Hara (2009, 2010), Epstein and Schneider (2010), and Cao, Han, Hirshleifer, and Zhang (2011). An exception is Epstein and Schneider (2008) who assume that investors are uncertain about the qualities of information signal. Such investors are averse to assets with poor information quality, which induces ambiguity premia. Watanabe (2016) provides an equilibrium setting in which investors are ambiguous about the mean of an asset's random supply shock. However, the focus of his paper is on market fragility.

In Li and Wang (2018), a representative investor faces model uncertainty about the financial market and so is uncertain about the composition of the efficient portfolio offered by a fund based on the fund's knowledge of the financial market. In their partial equilibrium setting where there is no risk sharing, this delegation uncertainty causes investor to delegate only partially, leading to higher CAPM alphas. In contrast, in our model, in equilibrium delegation uncertainty is endogenously eliminated by risk sharing among investors: holding one share of the *RAMP* fund in conjunction with the information-based portfolio is optimal.

II. The Financial Market

A continuum of investors with measure one who are indexed by i and uniformly distributed over $[0, 1]$ trade assets at date 0 and consume at date 1.

Assets. At date 0, each investor i can invest in a riskfree asset and $N \geq 2$ risky assets.[4] At date 1, the riskfree asset pays r units, and risky asset n pays f_n units of the single consumption good. In addition to trading directly, investors can also hold individual risky assets through a passive fund that commits to offering a portfolio X, which is an N-dimension column vector with the n^{th} element being the shares of the n^{th} risky asset in X.

Letting D_i be the vector of shares of the risky assets held by investor i, and d_i (a scalar) the shares of the fund held by investor i, investor i's effective risky asset holding is $d_i X + D_i$.

[4]To simplify notation, we assume that the number of risky assets N is common knowledge. However, our results hold in an extension where investors are also ambiguous about the total number of risky assets.

Return Information. Let $F = (f_1, f_2, \ldots, f_N)'$ be the vector of risky assets' returns. We assume that all investors share a common uniform improper prior of F, and so no investor has prior information about any risky asset's return.[5] Each investor i receives a vector of private signals S_i about asset returns, $S_i = F + \epsilon_i$, where F and ϵ_i are independent; and ϵ_i and ϵ_j are also independent. Each ϵ_i is normally distributed, with mean zero and precision matrix $\mathbf{\Omega}_i$. Since $\mathbf{\Omega}_i$ is the inverse of the variance-covariance matrix of ϵ_i, it is symmetric and positive definite. We do not require $\mathbf{\Omega}_i$ to be diagonal; hence, investor i's private signal about asset n's payoff may be correlated to that of asset k.

Letting $\mathbf{\Sigma}$ be the matrix of the average precision of private signals, we have

$$\mathbf{\Sigma} = \int_0^1 \mathbf{\Omega}_i \mathrm{d}i. \tag{1}$$

In addition, as is standard, the independence of the errors implies that in the economy as a whole investor private signal errors average to zero, so that the equilibrium pricing function does not depend on the error realizations (though it does depend on their distributions). Therefore, we have

$$\int_0^1 \mathbf{\Omega}_i S_i \mathrm{d}i = \mathbf{\Sigma} F. \tag{2}$$

Random Supply. Let Z denote the vector of random supply shocks of all risky assets. We assume that Z is independent of F and of ϵ_i (for all $i \in [0,1]$). We further assume that Z is normally distributed with mean 0 and the precision matrix $\boldsymbol{U}$. Here, $\boldsymbol{U}$ is also symmetric and positive definite.

[5] The uniform improper prior assumption is for simplicity. In Section A, we study the case in which investors have (potentially uncertain) normal priors of F. We show that with a modified portfolio offered by the passive fund, our results still hold.

Model Uncertainty and Ambiguity Aversion. Taking the riskfree asset to be the numeraire, let P be the price vector of the risky assets. Also, let $W_i = (w_{i1}, w_{i2}, \ldots, w_{iN})'$ be the endowed shareholdings of investor i. We assume that the aggregate endowments of shares of each stock are strictly positive; that is, $W = \int_0^1 W_i \mathrm{d}i \gg 0$. Then investor i's final wealth at date 1 is

$$\Pi_i = r\left[W_i' - \left(d_i X' + D_i'\right)\right] P + \left(d_i X' + D_i'\right) F. \tag{3}$$

The first term is the return of investor i's investment in the riskfree asset, and the second term is the total return from her investments in risky assets.

Since investor i is risk averse, if she knows all model parameters, at date 0 she maximizes a CARA expected utility function,

$$\mathbb{E}_i u(\Pi_i) = \mathbb{E}_i\left[-\exp\left(-\frac{\Pi_i}{\rho}\right)\right]. \tag{4}$$

The expectation in equation (4) is taken based on investor i's information about asset returns. Since the common prior about asset returns is uninformative, any investor i's information consists of the equilibrium price vector and the realization of a private signal S_i only.

However, investor i may be subject to model uncertainty. We assume that all parameters are common knowledge among investors, except the average private signal precision $\boldsymbol{\Sigma}$, the random supply shock precision $\boldsymbol{U}$, and the aggregate endowments of shares W. Therefore, we characterize a financial market, or a model, by $m = (\boldsymbol{\Sigma}, \boldsymbol{U}, W)$, and the set of all possible financial markets is denoted by $\mathcal{M}$. Let the nonempty set $\mathcal{M}_i \subseteq \mathcal{M}$ be the support of investor i's subjective belief over possible models. Then, when $\mathcal{M}_i$ is not a singleton, we say that investor i is subject to model uncertainty.[6]

We further assume that when investor i is subject to model uncertainty, she is ambiguity averse. Hence, she chooses an investment strategy (d_i, D_i) to maximize the infimum of her CARA utility. Formally, each investor i's decision problem is[7]

$$\max_{d_i, D_i} \inf_{m_i \in \mathcal{M}_i} \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho}\right)\right]. \tag{5}$$

The Passive Fund. We assume that the passive fund knows the true financial market parameters but does not have any private information about asset payoffs. Also, the fund does not know the model uncertainty each investor is facing or the fraction of investors who are subject to model uncertainties. Hence, the fund commits to offering the portfolio X based only on its knowledge of the model and the asset prices.[8]

Equilibrium. In our model, an investor i's investment strategy is a mapping from her subjective belief support $\mathcal{M}_i$, her private signals S_i, and asset prices P to the shares of assets she wants to hold

[6] We do not specify an ambiguity averse investor i's subjective belief over $\mathcal{M}_i$ and their subjective beliefs about other investors' subjective beliefs, since these will not affect our equilibrium characterization, as will become clear shortly.

[7] An investor's utility in this paper differs slightly from that defined in Gilboa and Schmeidler (1989). Since $\mathcal{M}_i$ may be a non-compact set, the investor maximizes the infimum rather than the minimum of her CARA utility among all possible models in $\mathcal{M}_i$.

[8] The fund does not trade actively as it has no private information about asset returns. In a setting where the fund has private information about asset returns, investors would have a further motive to delegate to the fund. The fund also behaves non-strategically in our model. Allowing for agency problems would add analytical complexity. However, an immediate implication of Proposition 5 is that if the passive fund wants to maximize the measure of investors who buy its shares, the optimal portfolio it commits to offering is the one specified in equation (13).

directly, D_i, and the shares of the fund she wants to hold, d_i. We are interested in an equilibrium defined as follows.

Definition 1. *A pricing vector P^* and a profile of all investors' risky asset holdings $\{d_i^*, D_i^*\}_{i\in[0,1]}$ constitute an equilibrium, if:*

1. *Given P^*, (d_i^*, D_i^*) solves investor i's maximization problem in equation (5), for all $i \in [0,1]$; and*

2. *P^* clears the market,*

$$\int_0^1 (d_i^* X + D_i^*)\,\mathrm{d}i = W + Z, \quad \text{for any realizations of } F \text{ and } Z. \tag{6}$$

III. Financial Market without Model Uncertainty

As a benchmark case for understanding the effects of model uncertainty, and for comparison of results with past literature, this section describes index investing and asset pricing in a financial market without model uncertainty. We assume that all investors commonly know the financial market parameters, so $\mathcal{M}_i = \{m\}$ for all $i \in [0,1]$. Therefore, any investor's max-min utility function degenerates to a regular CARA utility, and the model described in Section II becomes a traditional rational expectations equilibrium model, much like those studied by Admati (1985), Easley and O'hara (2004), Biais et al. (2010), and Andrei et al. (2023). We obtain new results in this setting that we then generalize to the case of model uncertainty.

A. Information Separation and Risk Adjusted Market Portfolio

Since the financial market parameters are commonly known among investors, and the passive fund does not have private information about asset returns, the index portfolio, X, offered by the passive fund can be constructed by the investors themselves. Hence, whether the index fund is available will not affect investors' equilibrium risky asset holdings. Following the literature, we characterize the equilibrium in this setting as follows.

Proposition 1. *In the model where the financial market parameters are all common knowledge among investors, there exists an equilibrium with the pricing function*

$$P = \boldsymbol{B}^{-1}\left[F - A - \boldsymbol{C}Z\right], \tag{7}$$

where

$$A = \frac{1}{\rho}\left[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}\right]^{-1} W \tag{8}$$

$$\boldsymbol{B} = r\boldsymbol{I} \tag{9}$$

$$\boldsymbol{C} = \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}. \tag{10}$$

Any investor i's total risky asset holding is

$$d_i X + D_i = \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Omega}_i\left(S_i - rP\right). \tag{11}$$

Equation (11) suggests that when the financial market parameters are common knowledge among investors, an investor's equilibrium holding of the index fund is undetermined; that is, there are multiple vectors of (d_i, D_i) that can lead to investor i's total risky asset holding characterized in equation (11). When the index fund offers the *VWMP* (denoted by $X_V = W$), assuming each investor holds one share of the index fund (i.e., $d_i = 1$), we can rewrite equation (11) as

$$d_i X + D_i = W + \left[\left\{ \left[\boldsymbol{I} + \frac{1}{\rho^2} \left(\boldsymbol{\Sigma U} \right)^{-1} \right]^{-1} - \boldsymbol{I} \right\} W + \rho \boldsymbol{\Omega}_i \left(S_i - rP \right) \right]. \tag{12}$$

Equation (12) represents a form of portfolio separation discussed in the literature. An investor holds the *VWMP* and tilts her portfolio towards stocks that she has information about. However, under information asymmetry, the index portfolio W is not optimal regardless of the investor's information signal. Therefore, the index fund offering W is not in general attractive to investors and thus does not facilitate index investing.

However, reexamination of equation (11) suggests another form of portfolio separation. As shown in the proof of Proposition 2, the first term in equation (11),

$$X_R = \left[\boldsymbol{I} + \frac{1}{\rho^2} \left(\boldsymbol{\Sigma U} \right)^{-1} \right]^{-1} W, \tag{13}$$

is the optimal portfolio conditional on public information only, while the second term in equation (11),

$$\rho \boldsymbol{\Omega}_i \left(S_i - rP \right), \tag{14}$$

is the optimal portfolio of investor i, conditional on her private signal S_i but not conditioned on the public information implicit in market price (and is also not conditioned on the financial market parameters). Since the prior distribution of asset returns is diffuse and no other public signals are available, public information here refers solely to the information that investors extract from equilibrium asset prices.

Crucially, the portfolio specified in equation (13) is independent of any investor's private information and so can be constructed based on the financial market parameters only. We dub such a portfolio the *Risk-Adjusted Market Portfolio* (*RAMP*), as it underweights more volatile assets relative to *VWMP*. Specifically, *RAMP* differs from *VWMP* in that it contains a component $(\mathbf{\Sigma U})^{-1}$. It then follows from equation (13) that *RAMP* includes fewer shares of more volatile assets. Therefore, holding *RAMP* is what practioners call a "defensive investment strategy," i.e., one that tends to reduce the risk of a large loss.

In contrast, the second component can be formed based only on the investor's own private information; it is independent of the information content of the market price and the financial market parameters. The reason for this independence is that the market is competitive. Each individual investor is small in the sense that her trading does not affect the asset prices and thus the price informativeness. Hence, we call portfolio $\rho\mathbf{\Omega}_i\left(S_i - rP\right)$ the *information-based portfolio* of investor i.

The separation described by equation (11) implies the Information Separation Theorem.

Proposition 2 (The Information Separation Theorem)**.** *When the financial market parameters are common knowledge, equilibrium portfolios have three components: a risk-adjusted market portfolio (RAMP); an information-based portfolio based upon private information and equilibrium prices but no extraction of information from prices; and the riskfree asset.*

Proposition 2 indicates that any investor can form an optimal portfolio in three separate steps: (1) buy one share of *RAMP*; (2) buy the information-based portfolio using only private information, not the information extracted from price; and (3) put any left-over funds into the riskfree asset. This separation theorem is a consequence of market equilibrium as well as optimization considerations. In this respect the information separation theorem differs from the non-informational separation theorems in the literature, which are based solely on individual optimization arguments.[9]

The Information Separation Theorem implies that if the index fund offers *RAMP*, instead of the *VWMP*, index investing becomes attractive even to investors who do not possess private information. In this sense, *RAMP* facilitates investors participation in the stock market via index investing. This benefit is even more important when investors are subject to model uncertainty, as analyzed in Section IV.

B. Asset Pricing with *RAMP* as Pricing Portfolio

Since the *VWMP* is not mean-variance efficient conditional on asset prices, risky assets will include non-zero risk premia relative to it. That is, with the *VWMP* being the pricing portfolio, risky assets will have non-zero alphas in a CAPM pricing relation.

Proposition 3. *In the setting where investors commonly know all financial market parameters but have asymmetric information about asset returns, with the* VWMP *being the pricing portfolio, risky assets have non-zero alphas.*

Proposition 3 implies that with the *VWMP* being the pricing portfolio, a CAPM pricing relation (conditional on public information) does not hold when investors hold asymmetric information.

[9]It may seem puzzling that none of the three portfolio components depend on the information that an investor extracts from price. How then does this information enter into the investor's portfolio decision? The answer is that *RAMP* is optimal precisely because of the ability of investors to extract information from price. Although *RAMP* is deterministic, the fact that *RAMP* is an optimal choice is true only because investors update their beliefs based on price.

This conclusion is supported by empirical studies, such as Fama and French (1992). Theoretical studies then derive different versions of CAPM pricing relation. For example, Easley and O'hara (2004) and Biais et al. (2010) obtain a version of CAPM pricing relation conditional on the "average" investor's private information using the total asset supply (i.e., the sum of endowed asset shares and random supply shocks) as the pricing portfolio. However, neither the "average" investor's belief nor the total asset supply is observable. Recently, Andrei et al. (2023) show that a version of CAPM pricing relation can hold with the *VWMP* as the pricing portfolio but needs a different definition of asset beta from that in the traditional CAPM.

We provide a different answer when revisiting the question of asset pricing under information asymmetry. The Information Separation Theorem indicates that *RAMP* may correctly price risky assets, since it is mean-variance efficient conditional on the public information. Formally, from equation (7), the equilibrium pricing function is

$$P = \frac{1}{r}\left[F - A - \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}Z\right], \tag{15}$$

where $A = \frac{1}{\rho}[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]^{-1}W$.

Let $\text{diag}(P)$ be an $N \times N$ diagonal matrix, whose n^{th} diagonal element is the n^{th} element of the vector P. Generically, as no asset has a zero price, $\text{diag}(P)$ is invertible. Then, by the definition of $\text{diag}(P)$, $\text{diag}(P)^{-1}P = \mathbb{1}$, where $\mathbb{1} = (1, 1, \ldots, 1)'$. We then have

$$\text{diag}(P)^{-1}\mathbb{E}(F|P) - r\mathbb{1} = \text{diag}(P)^{-1}A. \tag{16}$$

Here, $\mathbb{E}(F|P)$ is the expected payoff conditional on the equilibrium price. The LHS of equation (16) is just the vector of the risky assets' equilibrium risk premia.

Given a realized equilibrium price, *RAMP* has the value $P'X_R$. Then the vector of the weights of risky assets in *RAMP* is

$$\omega = \frac{1}{P'X_R}\text{diag}(P)X_R.$$

Hence, conditional on the price P, the difference between the expected return of *RAMP* and the riskfree rate is

$$\begin{aligned}
\mathbb{E}(R_{X_R}|P) - r &= \omega'\text{diag}(P)^{-1}\mathbb{E}(F|P) - r \\
&= \frac{1}{P'X_R}X_R'\text{diag}(P)\text{diag}(P)^{-1}(A + rP) - r \\
&= \frac{1}{P'X_R}X_R'A, \qquad (17)
\end{aligned}$$

where the expectations are all conditional on the equilibrium price. The variance of *RAMP* is

$$\mathbb{V}(R_{X_R}|P) = \mathbb{E}\left[\left(\omega'\text{diag}(P)^{-1}\boldsymbol{C}Z\right)\left(\omega'\text{diag}(P)^{-1}\boldsymbol{C}Z\right)'\right] = \left(\frac{1}{P'X_R}\right)^2 X_R'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R, \qquad (18)$$

and the covariance between all risky assets and *RAMP* is

$$Cov(R, R_{X_R}|P) = \frac{1}{P'X_R}\text{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R. \qquad (19)$$

Let α_R be the CAPM alpha with *RAMP* being the pricing portfolio. From equations (16)-(19), and since $X_R = \rho(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C})^{-1}A$, we have a new CAPM pricing relation.[10]

[10] Similar to the traditional CAPM, Proposition 4 presents a conditional (on price) CAPM pricing relation. It is

Proposition 4. *In the model with all parameters being common knowledge, conditional on asset prices, asset risk premia satisfy the CAPM security market line where the relevant market portfolio for pricing is RAMP. Formally, with RAMP being the pricing portfolio, assets' risk premia satisfy*

$$diag\,(P)^{-1}\,\mathbb{E}\,(F|P) - r\mathbb{1} = \beta_R\left[\mathbb{E}\left(R_{X_R}|P\right) - r\right], \tag{20}$$

where

$$\beta_R = \frac{1}{\mathbb{V}(R_{X_R}|P)}Cov(R, R_{X_R}|P). \tag{21}$$

This result may seem surprising, since investors have heterogeneous asset holdings, and since the portfolios held by informed investors are not mean-variance efficient with respect to public information. Nevertheless, in equilibrium, there are no extra risk premia incremental to those predicted by the CAPM using *RAMP*.

The CAPM pricing relation using *RAMP* is equivalent to the assertion that *RAMP* is mean-variance efficient conditional only on public information. This efficiency can be seen from the utility maximization problem of an investor who does not have any private signals about asset returns. Such an investor balances the expected returns and the risks of her holdings, and her information consists of the equilibrium price only, which is the only public information in our baseline model. In equilibrium, such an investor holds *RAMP*, implying that *RAMP* is mean-variance efficient conditional only on public information.

Privately informed investors also hold *RAMP* as the passive component of their portfolios. In addition, they have other asset holdings to take advantage of the greater safety of assets they have

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unclear whether an unconditional CAPM pricing relation will hold in our model because asset prices appear in the denominator in equation (20), making the equation nonlinear.

more information about, and for speculative reasons based upon their private information. *RAMP* is not mean-variance efficient with respect to their private information sets, but it is efficient with respect to the information set that contains only publicly available information.

IV. Financial Market Featuring Model Uncertainty

We have shown that in a financial market with information asymmetry but no model uncertainty, *RAMP* can facilitate index investing and can price risky assets where *VWMP* does not. We now show that these conclusions also apply in a financial market where investors are also subject to model uncertainty. Formally, in the model described in Section II, $\mathcal{M}_i$ may not be a singleton. In the face of such model uncertainty, ambiguity averse investors will choose portfolios that maximize their CARA expected utilities under worst case scenarios for the financial market parameters.

A. Index Investing under Ambiguity Aversion

When investors are ambiguous about the financial market parameters, they will eschew an index fund that offers the *VWMP*, $X_V = W$. To understand why, first observe that holding a positive position at the *VWMP* means holding a fixed positive position in each asset, since $W \gg 0$. When an investor is ambiguity averse about an asset and has no private information about its return, her posterior variance of the asset's return is unbounded in the worst case scenario. This means that holding a positive position at the *VWMP* implies an unboundedly negative expected CARA utility.

Therefore, the index fund that commits to offer the *VWMP* does not attract ambiguity averse investor.[11] Since this result is straightforward, we relegate the formal analysis to Appendix B.

This raises the question of whether the index portfolio designed in equation (13), *RAMP*, can facilitate index investing. We assume that although investors who are subject to model uncertainty do not know $\boldsymbol{\Sigma}$, $\boldsymbol{U}$, or W, and hence do not know the exact composition of X_R, the functional relationship between X_R and $(\boldsymbol{\Sigma}, \boldsymbol{U}, W)$ that is specified in equation (13) is common knowledge to investors.

Even though *RAMP* is offered based on the true parameters of the financial market, it is not obvious that an ambiguity averse investor will be willing to hold the fund. The investor does not know its exact composition, so holding it entails bearing extra risk. Furthermore, different investors may have different subjective beliefs about the structure of the financial market, which might make it seem unlikely that one market index can attract all investors. For example, when investor i and investor j completely disagree with each other about the financial market, i.e., $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$, it seems plausible that the optimal market index for investor i would differ from that for investor j.

However, we show in this subsection that with the passive fund offering *RAMP*, all investors employ the index investment strategy in equilibrium by holding one share of the fund, and their effective asset holdings are exactly the same as in the economy without model uncertainty. This is stated in Proposition 5.

Proposition 5. *In the model with a passive fund that commits to offering the portfolio X_R specified in equation* (13)*, there is an equilibrium in which*[12]

[11]In our model, the *VWMP*, $X_V = W$, may or may not be known to an investor. Regardless, if the investor believes that the average precision of private signals or the precision of the random supply shock can be zero, she will not hold *VWMP*.

[12]The equilibrium characterized in Proposition 5 is likely to be the unique linear equilibrium since it is the unique

1. *All investors will buy one share of the passive fund, and so* $d_i^* = 1$ *for all* $i \in [0,1]$*;*

2. *Any investor* i *will hold an extra portfolio* $D_i^* = \rho \boldsymbol{\Omega}_i (S_i - rP)$*; and*

3. *For any given* F *and* Z*, the equilibrium price is*

$$P = \boldsymbol{B}^{-1} [F - A - \boldsymbol{C} Z] , \tag{22}$$

where

$$A = \frac{1}{\rho} \left[\rho^2 (\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma} \right]^{-1} W \tag{23}$$

$$\boldsymbol{B} = r\boldsymbol{I} \tag{24}$$

$$\boldsymbol{C} = \frac{1}{\rho} \boldsymbol{\Sigma}^{-1}. \tag{25}$$

The intuition of Proposition 5 builds upon the Information Separation Theorem. We consider a strategy profile in which all investors hold one share of the passive fund and their own information-based portfolio. We will then argue that no investor has an incentive to deviate unilaterally from such a strategy.

To build intuition, we first consider investors with min-max preferences, and then show that our conclusions also apply to max-min preferences as well. With min-max preferences, we argue that given other investors' strategies of holding *RAMP* and their own information-based portfolios, an investor's optimal trading strategy is constant across all possible values of financial market

linear equilibrium in the model without model uncertainty. However, as is typical of rational expectations equilibrium models, here (even with *RAMP* being offered) it is rather intractable to show the equilibrium uniqueness.

parameters in her subjective belief support. Hence, the investor's optimal trading strategy with max-min utility is the same as that with min-max utility.

Specifically, when investor i has a min-max utility, for each possible world $m_i \in \mathcal{M}_i$, she can solve her optimal risky asset holdings, assuming that the equilibrium pricing function is the one in equation (22) with m being m_i. Importantly, because all other investors are holding one share of the fund along with their own direct information-based portfolios, they are effectively holding the risky assets as in the world with m_i being common knowledge. Therefore, in the possible world m_i, the market clearing condition implies that the pricing function is the one specified in equation (22) with m being m_i. That is, investor i's belief about the pricing function is correct. So, she would like to hold the risky assets as in the world m_i. Such risky asset holdings can be implemented by holding one share of the passive fund and her information-based portfolio, so investor i would like to use the investment strategy in Proposition 5. Furthermore, investor i is still uncertain about m, so holding the risk-adjusted market portfolio through holding one share of the fund is strictly preferred.

In the above, for any given possible world, holding one share of the fund and her own information-based portfolio maximizes investor i's expected CARA utility (given that all other investors trade according to the prescribed strategy profile). Since such a trading strategy is optimal across all possible values of financial market parameters, it maximizes investor i's max-min utility. That is, a strong max-min property holds in the equilibrium, and hence, in our model with investors having max-min utilities, the investment strategy of holding one share of the fund and the information-based portfolio is also optimal to investors.

So the Information Separation Theorem helps explain why ambiguity averse investors are willing to hold the fund that offers *RAMP* in equilibrium. Indeed, the same argument can also be applied when investors have the smooth ambiguity preferences (Klibanoff, Marinacci, and Mukerji, 2005). Since holding one share of the passive fund and her own information-based portfolio helps an investor to achieve the optimal balance between risk and return in each possible model in her subjective belief support, the optimal investment strategy will be independent of the utility representations of ambiguity preference.

B. Properties of the Equilibrium

The equilibrium characterized in Proposition 5 has several interesting properties describing behavior when a *RAMP* index fund is available. First, investors all hold exactly one share of the fund, even though they have heterogeneous priors about the financial market and thus different beliefs about the fund's composition.[13] This is true even in the extreme case in which two investors, i and j, completely disagree about the financial market parameters; that is, $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$. By Proposition 5, both investor i and investor j hold one share of the passive fund, along with their own information-based portfolios. So differences in investors' holdings arise solely from differences in their private signals about asset payoffs, not from differences in their model uncertainties.

Second, *RAMP* does not solve the problem of enabling market participation by eliminating model uncertainty; it solves the problem by providing an attractive vehicle for risk sharing.[14] Proposition 5 shows that the willingness of investors to buy an index is based on an understanding of equilibrium risk-sharing, rather than just a partial equilibrium understanding that there can be

[13]For simplicity in presentation, we focus on the setting where investors have homogeneous risk preferences and thus optimally hold one share of the passive fund. In Section V, we show that when investors have heterogeneous risk preferences, they will hold different amounts of the passive fund. However, the main results of our model continue to hold in this extended setting.

[14]Mele and Sangiorgi (2015) show that in a model without a passive fund offering *RAMP*, ambiguity averse investors acquire costly information about model parameters, which helps reduce model uncertainty.

risk-reduction benefits to the investor in isolation to diversifying her portfolio. Specifically, consider an investor who faces model uncertainty about a subset of traded assets, and views the return distributions as exogenous. Even if she can indirectly trade those assets through a passive fund, it may not be optimal for her to do so, because she cannot calculate the fund's expected return and risk.

It follows that arguments based on the incentive of individuals to diversify do not, under radical ignorance, justify holding an index fund. In contrast, in our equilibrium setting, an investor optimally holds the fund, given her belief that other investors will also do so (together with their direct portfolios). So she is willing to hold the fund too, which achieves the benefit of optimally sharing risks with other investors.[15]

Proposition 5 more broadly suggests that the reason why actual investors often fail to diversify goes beyond investor ambiguity aversion. In particular, for an investor to hold the fund, all other investors need to behave according to the prescribed equilibrium strategy profile. If imperfectly rational investors reason about possible portfolios based solely on partial equilibrium risk and return arguments, portfolios containing assets that investors are ambiguous about might seem extremely risky (or in the limiting case, infinitely risky). Proposition 5 shows that, owing to equilibrium considerations, even ambiguity averse investors, if otherwise rational, will hold such assets. But actual investors may not understand the equilibrium reasoning which underlies this result.

[15]The fact that equilibrium rather than just diversification considerations are crucial for the index investment result can be seen more concretely by considering the off-equilibrium possibility that other investors trade in a fashion that causes asset prices to be almost uninformative. In such a scenario, an ambiguity averse investor would not hold the passive fund, because *RAMP* would be perceived as extremely risky.

C. Asset Pricing under Ambiguity Aversion

Finally, we study asset pricing when investors are ambiguity averse. Consider first the CAPM pricing relation using the *VWMP* as the pricing portfolio. For simplicity, we assume that if investor i is ambiguity averse about asset n, then she does not have private information about asset n's return either (i.e., the n^{th} diagonal element of $\mathbf{\Omega}_i$ is zero). Lemma 1 in Appendix B then shows that investor i will hold a zero position of asset n. We further assume that for asset n, there are $1 - q_n$ measure investors who are ambiguity averse about asset n. Proposition 9 in Appendix B characterizes an equilibrium when investors are ambiguity averse and the passive fund commits to offer the *VWMP*. In such an equilibrium, the pricing function is

$$P = \boldsymbol{B}_V^{-1} \left[F - A_V - \boldsymbol{C}_V Z \right], \tag{26}$$

where

$$A_V = \frac{1}{\rho} \left[\rho^2 (\mathbf{\Sigma} \boldsymbol{Q} \boldsymbol{U} \mathbf{\Sigma}) + \mathbf{\Sigma} \right]^{-1} W \tag{27}$$

$$\boldsymbol{B}_V = r\boldsymbol{I} \tag{28}$$

$$\boldsymbol{C}_V = \frac{1}{\rho} \mathbf{\Sigma}^{-1}, \tag{29}$$

and $\boldsymbol{Q}$ is the $N \times N$ diagonal matrix with the n^{th} diagonal entry being q_n.

Given the asset prices, we calculate the capitalization weights of assets in the *VWMP*, the variance of the *VWMP*, the covariance between risky assets and the *VWMP*, and then the asset betas. Consequently, we derive assets' alphas, using the *VWMP* as the pricing portfolio, as

$$\alpha_V = \text{diag}(P)^{-1} \left[\boldsymbol{I} - \frac{\boldsymbol{C}_V \boldsymbol{U}^{-1} \boldsymbol{C}_V W W'}{W' \boldsymbol{C}_V \boldsymbol{U}^{-1} \boldsymbol{C}_V W} \right] A_V. \tag{30}$$

Proposition 6 shows that when investors hold asymmetric information and are ambiguity averse, the CAPM pricing relation does not hold with the *VWMP* as the pricing portfolio.

Proposition 6. *With the VWMP being the pricing portfolio, risky assets have non-zero alphas.*

The non-zero alphas show that in contrast with the traditional CAPM, in our model, the *VWMP* does not price assets correctly in the cross section. These alphas derive from both information asymmetry and ambiguity aversion. First, the traditional CAPM is based on homogeneous beliefs which are fully impounded in the market capitalization weights in *VWMP*. Hence, conditional on the asset prices, *VWMP* is mean-variance efficient. In contrast, in our setting, owing to information asymmetry, the average precision of private signals ($\boldsymbol{\Sigma}$) and the precision of random supply ($\boldsymbol{U}$) will determine the price signal distribution. This is directly implied by the equilibrium pricing function (equation (54)). Therefore, the weights in a portfolio that can price assets correctly must be functions of these two precisions. The *VWMP*, however, has value weights, which are not functions of these two precisions, so it cannot be efficient conditional on asset prices in the financial market with information asymmetry. Hence, using *VWMP* as the pricing portfolio, the assets should have non-zero alphas.

Second, ambiguity aversion also affects appropriate index portfolio weights for asset pricing, which further contributes to nonzero alphas relative to *VWMP*. Intuitively, when more investors are ambiguous about an asset, its demand curve shifts leftward, leading to a lower price and a higher risk premium. We then refer to the increment in the asset's risk premium due to ambiguity aversion as the asset's *ambiguity premium*.

We now consider a CAPM pricing relation with *RAMP* being the pricing portfolio. Proposition 5 indicates that the model with ambiguity aversion and a fund that commits to offer *RAMP* has an equilibrium in which investors' effective risky assets holdings are exactly the same as in the rational expectations equilibrium without model uncertainty. This suggests that assets' ambiguity premia should disappear. In addition, as shown in Proposition 4, with *RAMP* being the pricing portfolio, any risky asset has a zero alpha. This follows from the Information Separation Theorem: *RAMP* is the efficient portfolio conditional only on the price signals. Hence, given that the ambiguity premia will be eliminated by the passive fund that commits to offer *RAMP*, a version of CAPM security market line will hold under information asymmetry and ambiguity aversion, with *RAMP* as the benchmark pricing portfolio. Corollary 1 presents this surprising result.

Corollary 1. *In the model where investors are subject to model uncertainty, if a passive fund offers RAMP as specified in equation* (13)*, asset risk premia satisfy the CAPM with RAMP as the pricing portfolio.*

Corollary 1 implies that *RAMP* outperforms *VWMP* in a single-factor model. That is, when we replace *VWMP* with a proxy of *RAMP* as the common factor in the single-factor model, the average absolute value of estimated asset alphas will decrease. We perform preliminary tests of this prediction by using a traded low-volatility ETF (the Invesco S&P 500® Low Volatility ETF that tracks the S&P 500® Low Volatility Index) as our proxy for *RAMP*.

The S&P 500 Low Volatility Index shares the key features of *RAMP* of weighting assets positively by market values and negatively by volatilities. As introduced by S&P Indices (2011), the index is constructed in the following three steps. First, the index committee calculates the historical volatilities of the 500 stocks in the S&P 500 Index using daily standard deviations. Second, stocks are ranked in ascending order of their historical volatilities. Finally, positive weights are placed only on the 100 least volatile stocks, with the weight on each stock proportional to the reciprocal of its volatility.

The empirical results are consistent with Proposition 6 and Corollary 1. Using the Invesco S&P 500® Low Volatility ETF (as a proxy for *RAMP*) as the common factor, the average absolute value of estimated asset alphas is lower than when using the *VWMP* as the common factor. With the Low Volatility ETF, this mean becomes statistically insignificant. We present the tests and the empirical results in Online Appendix A.

However, several caveats are needed to interpret this empirical test. In general, *RAMP* differs from the S&P 500 Low Volatility Index in several respects. First, the weights in *RAMP* depend on more than volatility. In general, they also depend upon risk aversion, the correlation of signals across assets for each investor, and correlated random supply shocks. Hence, for the S&P 500 Low Volatility Index to serve as a good proxy for *RAMP*, both investors' private information precision matrices and the random supply shock precision matrix must be diagonal. Second, even when all assets are independent, the asset weights in *RAMP* are a strictly increasing but nonlinear function of the reciprocal of asset volatilities. In contrast, the S&P 500 Low Volatility Index assigns weights based solely on the reciprocal of volatilities. As a result, the relationship between asset weights in *RAMP* and those in the S&P 500 Low Volatility Index is inherently nonlinear. Owing to this nonlinearity, an ambiguity-averse investor cannot replicate *RAMP* by combining the S&P 500 Low Volatility Index with other assets. Additionally, the S&P 500 Low Volatility Index consists of only the 100 stocks with the lowest volatility, further limiting its ability to approximate *RAMP*.

Due to these differences, the Invesco S&P 500® Low Volatility ETF that tracks the S&P 500 Low Volatility Index — despite being publicly available — may not effectively encourage ambiguity averse investors to participate in the financial market.

Although the S&P 500 Low Volatility Index is an imperfect proxy for *RAMP*, the empirical results presented in Online Appendix A suggest that a well-designed low volatility index can encourage ambiguity averse investors to participate in financial markets and thereby reduce pricing errors. In other words, a carefully constructed low volatility index can act as a stepping stone toward providing *RAMP*.

Accordingly, in Online Appendix B, we discuss in detail the relationship between *RAMP* and low volatility indexes. We show that a low volatility index that is designed to be a component of *RAMP* can attract ambiguity averse investors to trade the stocks included in the index, thereby reducing pricing errors — though these investors may still refrain from trading assets outside the index. Our numerical simulations further demonstrate that the stronger is investor ambiguity aversion, the more inclusion of the constructed low volatility index reduces pricing errors. This underscores the potential role of a well-constructed low volatility index as an important stepping stone toward implementing *RAMP*.

V. Extensions

We now consider the robustness of our conclusions to allowing for a Gaussian prior or for heterogeneous risk aversions.

A. Normal Prior

In the model of Section II, we assume for simplicity that investors hold a common uniform improper prior. We next verify that similar conclusions hold when we consider ambiguity averse investors' asset holdings with normal priors about asset payoffs. Arguably, this is a more realistic scenario.

Formally, we assume that the asset payoffs are drawn from the distribution $F \sim \mathcal{N}\left(\overline{F}, \boldsymbol{V}^{-1}\right)$, where $\overline{F}$ is the ex-ante mean and $\boldsymbol{V}$ is the ex-ante precision matrix. Investors, however, may be ambiguous about such a distribution. So, in this case, a model should be characterized by $m = (\boldsymbol{\Sigma}, \boldsymbol{U}, W, \overline{F}, \boldsymbol{V})$.

We assume that in this case, the passive fund commits to offering a portfolio

$$Y = \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\right]^{-1} V\left(\overline{F} - rP\right). \tag{31}$$

The portfolio Y is the sum of *RAMP* and a portfolio for contrarian trading. It can be shown to be the optimal portfolio in the setting without model uncertainty based on the prior and the price signal. However, since the prior and the price signal are not conditionally independent (because the prior is a determinant of the price signal), the two components that constitute the portfolio Y are not information-separated.

Proposition 7 below shows that with the fund offering the portfolio Y, all investors will hold exactly one share of the fund, and any investor's overall risky asset holding is the same as that in the setting without model uncertainty.

Proposition 7. *In the model with normal priors, when there is a passive fund that commits to offering the portfolio Y specified in equation (31), there is an equilibrium in which*

1. *All investors will buy one share of the passive fund, and so $d_i^* = 1$ for all $i \in [0,1]$; and*

2. *Any investor i will hold an extra portfolio $D_i^* = \rho \mathbf{\Omega}_i (S_i - rP)$.*

We further analyze asset risk premia with the portfolio Y being the pricing portfolio. While it is rather intractable to show that any asset's alpha is zero, Corollary 2 below argues that the portfolio Y is the optimal portfolio choice of investors who do not have any private information. Since investors with no private information choose their portfolios based only on all public information (in this case, the prior and the price signal), Y is mean-variance efficient conditional on all public signals.

Corollary 2. *In the model with normal priors, when there is a passive fund that commits to offering the portfolio Y, any investor i with $\mathbf{\Omega}_i = \mathbf{0}$ will choose the portfolio Y in equilibrium.*

B. Heterogeneous Risk Aversions

In the model of Section II, investors share a same risk aversion coefficient ρ. This leads to homogeneous holdings by investors of the passive fund. Indeed, in the equilibrium characterized in Proposition 5, all investors hold one share of the passive fund. However, this raises the question of whether investors with different risk tolerances are willing to hold a single common passive fund, and if so, whether differences in risk tolerances, and investor unawareness of other investors' risk tolerances might result in heterogeneous holdings of the passive fund. To evaluate the robustness of our conclusions, we now extend the model to allow for heterogeneous risk tolerances.

We extend the model in Section II by assuming that any investor i ($i \in [0,1]$) has the risk aversion coefficient ρ_i. Here, ρ_i is a continuous function of i. Let

$$\overline{\rho} = \int_0^1 \rho_i \mathrm{d}i \quad \text{and} \quad \overline{\boldsymbol{\Sigma}} = \int_0^1 \rho_i \boldsymbol{\Omega}_i \mathrm{d}i.$$

Here, $\overline{\rho}$ is the average risk tolerance, and $\overline{\boldsymbol{\Sigma}}$ is the average precision of investors' private information that is weighted by their risk tolerances. We assume that any investor i knows ρ_i, but she does not know the distribution of ρ_j or the average risk tolerance $\overline{\rho}$. Therefore, a financial market is characterized by $m = (\overline{\boldsymbol{\Sigma}}, \boldsymbol{U}, W, \overline{\rho})$. The passive fund cannot evaluate each individual investor's risk tolerance, but it has accurate information about the distribution of investors' risk tolerances; hence, it knows $\overline{\rho}$ and $\overline{\boldsymbol{\Sigma}}$. Then, the passive fund offers the portfolio

$$\overline{\mathrm{X}} = \left[\overline{\rho}\boldsymbol{I} + \left(\overline{\boldsymbol{\Sigma}}\boldsymbol{U}\right)^{-1}\right]^{-1} W. \tag{32}$$

to all investors. Proposition 8 characterizes equilibrium portfolios in such an extension.

Proposition 8. *In the model with heterogeneous risk tolerances, when there is a passive fund that commits to offering the portfolio $\overline{\mathrm{X}}$ specified in equation (32), there is an equilibrium in which*

1. *All investors will buy ρ_i share of the passive fund, and so $d_i^* = \rho_i$ for all $i \in [0,1]$; and*

2. *Any investor i will hold an extra portfolio $D_i^* = \rho_i \boldsymbol{\Omega}_i \left(S_i - rP\right)$.*

It directly follows from Proposition 8 that with a passive fund offering the portfolio $\overline{\mathrm{X}}$, investors hold the same portfolios as they do when they are not facing model uncertainty. Therefore, the conclusion of our baseline model that a wisely designed index can encourage investors to participate in the financial market and engage in index investing is robust.

On the other hand, Proposition 8 shows that the number of shares of the passive fund an investor holds depends on her risk tolerance. This is similar to that in the classic CAPM — all investors invest in the same index but the shares they hold the index fund depend on their risk tolerance.

VI. Concluding Remarks

We study two major roles played by market indexes — facilitating diversified investing, and providing an appropriate asset pricing benchmark — in a financial market with information asymmetry, model uncertainty, and ambiguity aversion. When investors possess heterogeneous information, offering the Value-Weighted Market Portfolio (*VWMP*) as an index does not encourage investors to engage in index investment. The *VWMP* is not generically mean-variance efficient conditional on either private or public information signals. As a result, risky assets have non-zero alphas relative to the *VWMP* as the pricing portfolio in a CAPM pricing relation.

If investors are also subject to model uncertainty — that is, they do not know some financial market parameters — then ambiguity averse investors will still refrain from holding the *VWMP*, even if a passive fund commits to offering it. In such a case, the alphas of risky assets in the CAPM pricing relation include ambiguity premia.

We introduce a new market index that can replace the traditional roles of the *VWMP* in the classic CAPM in investor portfolio formation and the pricing of risky assets. Such an index adjusts market value weights to take into account the average precision of investor private signals and the precision of random supply of different assets, i.e., the amount of risk reduction investors obtain by conditioning on price as a signal. We call this index the *Risk-Adjusted Market Portfolio* (*RAMP*). *RAMP* is a defensive strategy in the sense that, relative to the value-weighted market portfolio, it underweights assets that are more volatile. It can also be viewed as a smart-beta investment strategy, an approach that has gained great popularity in recent years.

In our setting, the ability of investors to invest in a *RAMP* index fund has major implications for equilibrium trading and asset pricing. In equilibrium, regardless of heterogeneity in their subjective beliefs about the financial market, all investors hold the index as the passive component of their portfolios. That is, *RAMP* induces investors to diversify by employing an index investment strategy. This improves the sharing of risk between investors who face model uncertainty about an asset and those who do not. In equilibrium, all investors' asset holdings are exactly the same as those in the economy without model uncertainty. Finally, the CAPM pricing relationship holds with respect to *RAMP* as the benchmark pricing portfolio, even though investors have asymmetric information and face model uncertainty. These results imply that delegation is a potential way to alleviate the inefficiency caused by model uncertainty.

These properties of *RAMP* derive from an Information Separation Theorem in the financial market without model uncertainty. The Information Separation Theorem says that to attain her optimal asset holdings, an investor first constructs an optimal portfolio based on each of her signals (i.e., price signal and private signal) and then sums all these optimal portfolios together. Then, in the setting with model uncertainty, when other investors are holding a passive fund offering *RAMP* and their information-based portfolios, in any possible financial market in her subjective belief support, an investor's optimal investment strategy is also to hold the fund and her own information-based portfolio. Therefore, providing *RAMP* to all investors facilitates their asset market participation and risk sharing.

Since *RAMP* underweights high volatility stocks, it is a defensive investing strategy. The investment strategy of following *RAMP* can also be viewed as a smart beta strategy, an approach that has been gaining popularity in practice. Furthermore, *RAMP* reduces equity premia, so a tradable *RAMP* index mitigates the equity premium puzzle. Moreover, it is likely that investors face even more severe model uncertainty about foreign countries' financial markets, which has been offered as an explanation for home bias in investment. This suggests that an international *RAMP* fund could help address home bias.

Since the *RAMP* fund can help promote investor participation and risk sharing, policy makers can consider facilitating the offering of funds that approximate it. This may be especially relevant in countries with less developed financial markets, where retail investors may be more likely to lack the information needed for estimating the relevant parameter values of *RAMP*. For example, policymakers may collect and provide information for constructing *RAMP*. To make use of distributed expertise, a policymaker may design a mechanism so that even if no manager individually has sufficient knowledge to construct *RAMP*, it can be implemented by a fund of funds, where each constituent fund is run by a manager with partial knowledge of the unknown parameters. We provide a formal analysis about such a mechanism design problem in Online Appendix C.

Finally, our analysis suggests a new reason why there may be value to informing investors about concepts of optimal risk sharing. For *RAMP* to play its role, investors need to trade as prescribed in equilibrium. This may prevent the implementation of *RAMP*. This challenge is similar to that faced in introducing products with network externalities such as telephone systems, email, and social media. A potential stepping stone (or on-ramp) toward implementing *RAMP* would be the introduction of low-volatility portfolios that are *RAMP* components.[16] Also, effective finan-

[16]It is not clear that such low-volatility portfolios can already be constructed using existing low-volatility indexes. Evaluating this would require estimation of the parameters in *RAMP* and examine their relation to detailed institutional information about the portfolio weights of those indexes.

cial education programs may facilitate investors' coordination and thereby encourage investors to engage in *RAMP* investing.

A. Proofs

In this appendix, we present proofs of all propositions and lemmas in the paper.

Proof of Proposition 1:

Let's first prove a more general version of Proposition 1, when investors hold a common prior belief about F, $F \sim \mathcal{N}\left(\overline{F}, \boldsymbol{V}\right)$. As is standard in the literature of rational expectations equilibrium, we consider the linear pricing function

$$F = A + \boldsymbol{B}P + \boldsymbol{C}Z, \quad \text{with } C \text{ nonsingular.} \tag{33}$$

If and only if B is nonsingular, equation (33) can be rearranged to

$$P = -\boldsymbol{B}^{-1}A + \boldsymbol{B}^{-1}F - \boldsymbol{B}^{-1}\boldsymbol{C}Z, \tag{34}$$

which solves for prices. Recall that $S_i = F + \epsilon_i$, so conditional on F, P and S_i are independent. Therefore, we can write down assets' payoffs' posterior means and posterior variances conditional on all information that are available to investor i as follows.

First consider investor i's belief about F conditional on P. Conditional on P, F is normally distributed with mean $A + \boldsymbol{B}P$ and precision $[\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}']^{-1}$. On the other hand, conditional on S_i, investor i's belief about F is also normally distributed, with mean S_i and precision $\boldsymbol{\Omega}_i$. Therefore, investor i's belief about F conditional on what the investor observes, P and S_i, is also normally distributed. The mean of the conditional distribution of F is the weighted average of the expectation conditional on the price P, the expectation conditional on investor i's private signal S_i, and the prior mean $\overline{F}$. Therefore, the conditional mean of F is

$$\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right]^{-1} \left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right]. \tag{35}$$

The precision of the conditional distribution of F is

$$(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}. \tag{36}$$

Then, from any investor i's first order condition, investor i's demand is

$$\begin{aligned}
D_i &= \rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right] \\
&\quad \left\{\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right]^{-1}\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right] - rP\right\} \\
&= \rho\left\{\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right] - \left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right] rP\right\} \\
&= \rho\left\{(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\boldsymbol{\Omega}_i - r\boldsymbol{V}^{-1}\right\} P \\
&\qquad + \rho\boldsymbol{\Omega}_i S_i + \rho[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}].
\end{aligned} \tag{37}$$

Integrating across all investors' demands gives the aggregated demand as

$$\begin{aligned}
\int_0^1 D_i \mathrm{d}i = &\rho\left\{(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\left(\int_0^1 \boldsymbol{\Omega}_i \mathrm{d}i\right) - r\boldsymbol{V}^{-1}\right\} P \\
&+ \rho\left(\int_0^1 \boldsymbol{\Omega}_i S_i \mathrm{d}i\right) + \rho[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}].
\end{aligned} \tag{38}$$

By equation (1), we have $\int_0^1 \boldsymbol{\Omega}_i \mathrm{d}i = \boldsymbol{\Sigma}$. Also, note that

$$\int_0^1 \boldsymbol{\Omega}_i S_i \mathrm{d}i = \boldsymbol{\Sigma} F.$$

Therefore, from the market clearing condition, we have

$$\int_0^1 D_i \mathrm{d}i = Z + W. \tag{39}$$

In an equilibrium, both equation (33) and equation (39) hold simultaneously for any realized F and Z, therefore, by matching coefficients in these two equations, we have

$$\rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}\right] - W = -\boldsymbol{C}^{-1}A \tag{40}$$

$$\rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\boldsymbol{\Sigma} - r\boldsymbol{V}^{-1}\right] = -\boldsymbol{C}^{-1}\boldsymbol{B} \tag{41}$$

$$\rho\boldsymbol{\Sigma} = \boldsymbol{C}^{-1} \tag{42}$$

Therefore, from equation (42), we have

$$\boldsymbol{C} = \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}$$

Obviously, $\boldsymbol{C}$ is positive definite and symmetric. Then from equation (40), we have

$$[\rho^2(\boldsymbol{\Sigma}\boldsymbol{U}\boldsymbol{\Sigma}) + \boldsymbol{\Sigma}]A = \frac{1}{\rho}W - \boldsymbol{V}^{-1}\overline{F}.$$

Because both $(\boldsymbol{\Sigma U \Sigma})$ and $\boldsymbol{\Sigma}$ are both positive definite, we have

$$A = [\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]^{-1} \left(\frac{1}{\rho} W - \boldsymbol{V}^{-1} \overline{F} \right).$$

From equation (41), we have

$$[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}](\boldsymbol{B} - r\boldsymbol{I}) = r\boldsymbol{V}^{-1}.$$

Again, because $[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]$ is positive definite, we have

$$\boldsymbol{B} = r\boldsymbol{I} + r[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]^{-1}\boldsymbol{V}^{-1}.$$

Obviously, $\boldsymbol{B}$ is invertible. By substituting A, $\boldsymbol{B}$, and $\boldsymbol{C}$ into equation (34), we solve the equilibrium pricing function.

Now, let's look at any investor i's holding. Substituting the coefficients into investor i's holding function (37), we have

$$D_i = \left(\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma U})^{-1} \right)^{-1} W + \rho \left[\boldsymbol{I} + \rho^2 \boldsymbol{\Sigma U} \right]^{-1} \boldsymbol{V}^{-1}(\overline{F} - rP) + \rho \boldsymbol{\Omega}_i (S_i - rP).$$

Finally, because the pricing function P and any investor i's demand function D_i are continuous in $\boldsymbol{V}^{-1}$, we can substitute $\boldsymbol{V}^{-1} = 0$ to get Proposition 1.

Q.E.D.

Proof of Proposition 2:

From Proposition 1, an investor's belief about asset returns conditional on the stock price is

$$F|P \sim \mathcal{N}\left(\frac{1}{\rho}\left[\rho^2(\mathbf{\Sigma U \Sigma}) + \mathbf{\Sigma}\right]^{-1} W + rP, \frac{1}{\rho^2}\left[\mathbf{\Sigma U \Sigma}\right]^{-1}\right).$$

Hence, conditional on the stock price only, an investor's optimal portfolio holding is

$$\rho \mathbb{V}\left(F|P\right)^{-1}\left(\mathbb{E}\left(F|P\right) - rP\right) = \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\mathbf{\Sigma U}\right)^{-1}\right] W,$$

which is the first term of equation (11), or *RAMP*.

Similarly, conditional on her private information S_i, investor i's belief about asset returns is

$$F|S_i \sim \mathcal{N}\left(S_i, \mathbf{\Omega}_i^{-1}\right).$$

Hence, conditional on the stock price only, an investor's optimal portfolio holding is

$$\rho \mathbb{V}\left(F|S_i\right)^{-1}\left(\mathbb{E}\left(F|S_i\right) - rP\right) = \rho \mathbf{\Omega}_i\left(S_i - rP\right),$$

which is the second term of equation (11), or investor i's information-based portfolio.

Since investor i's equilibrium portfolio based on both the price and her private signal is just the sum of these two portfolios, the Information Separation Theorem holds.

Q.E.D.

Proof of Proposition 3:

This is a special case of the general model where investors not only have asymmetric information but also are subject to model uncertainties. Hence, the proof is also a special case of that for Proposition 6.

Proof of Proposition 4:

By equations (17), (18), and (19), we have the right-hand side of equation (20) as

$$\begin{aligned} & \beta_R \left[\mathbb{E}\left(R_{X_R}|P\right) - r\right] \\ = \; & \frac{1}{\mathbb{V}(R_{X_R}|P)} Cov(R, R_{X_R}|P) \left[\mathbb{E}\left(R_{X_R}|P\right) - r\right] \\ = \; & \frac{\frac{1}{P'X_R}\mathrm{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R}{\left(\frac{1}{P'X_R}\right)^2 X_R'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R} \frac{X_R'A}{P'X_R} \\ = \; & \frac{\mathrm{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R}{X_R'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R} X_R'A. \end{aligned} \tag{43}$$

We want to show that equation (43) equals $\mathrm{diag}\left(P\right)^{-1}\mathbb{E}\left(R|P\right) - r\mathbb{1}$, which is the left-hand side of equation (20). Equation (16) shows that

$$\mathrm{diag}\left(P\right)^{-1}\mathbb{E}\left(R|P\right) - r\mathbb{1} = \mathrm{diag}(P)^{-1}A.$$

Hence, we just need to show

$$\frac{\mathrm{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R}{X_R'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R} X_R'A = \mathrm{diag}(P)^{-1}A.$$

Note that because $X_R = \rho(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C})^{-1}A$, and $(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C})^{-1}$ is a symmetric matrix, we have

$$\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R X_R'A = AX_R'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R, \tag{44}$$

which implies

$$\mathrm{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R X_R' A = \mathrm{diag}(P)^{-1} A X_R' \boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R.$$

Therefore, it follows that

$$\frac{\mathrm{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R}{X_R'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X_R} X_R' A = \mathrm{diag}(P)^{-1} A,$$

which shows that equation (20) holds.

Q.E.D.

Proof of Proposition 5:

We first verify that the market clearing condition holds. Each investor i's effective risky assets holding is

$$d_i^* X + D_i^* = \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Omega}_i\left(S_i - rP\right).$$

Then, using the pricing function (equation (22)), the aggregate demand can be calculated as

$$\begin{aligned}
& \int_0^1 \left(d_i^* X + D_i^*\right) \mathrm{d}i \\
= \; & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Sigma}\left(F - rP\right) \\
= \; & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Sigma}\left(\frac{1}{\rho}\left(\boldsymbol{\Sigma} + \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\boldsymbol{\Sigma}\right)^{-1} W + \frac{1}{\rho}\boldsymbol{\Sigma}^{-1} Z\right) \\
= \; & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\right]^{-1} W + Z \\
= \; & \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\right]^{-1} W + \left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\right]^{-1} W + Z \\
= \; & W + Z.
\end{aligned}$$

Therefore, the market clears.

We then show that when all other investors choose the prescribed investment strategies, any investor i will not make a unilateral deviation from the prescribed investment strategy either. That is, if $d_j^* = 1$ and $D_j^* = \rho \boldsymbol{\Omega}_j \left(S_j - rP \right)$ for all $j \in [0,1] \setminus \{i\}$, then $d_i^* = 1$ and $D_i^* = \rho \boldsymbol{\Omega}_i \left(S_i - rP \right)$.

In our model, any investor understands that she is small and so her trading will not affect the aggregate demand. Hence, for a fixed financial market $m_i \in \mathcal{M}_i$, investor i knows that all other investors' investment strategies lead to the aggregate asset demand

$$
\begin{aligned}
& \int_0^1 \left(d_j^* X + D_j^* \right) \mathrm{d}j \\
= & \left[\boldsymbol{I} + \frac{1}{\rho^2} \left(\boldsymbol{\Sigma} \boldsymbol{U} \right)^{-1} \right]^{-1} W + \int_0^1 \rho \boldsymbol{\Omega}_j \left(S_j - rP \right) \mathrm{d}j \\
= & \left[\boldsymbol{I} + \frac{1}{\rho^2} \left(\boldsymbol{\Sigma} \boldsymbol{U} \right)^{-1} \right]^{-1} W + \rho \boldsymbol{\Sigma} F - \rho r \boldsymbol{\Sigma} P.
\end{aligned}
$$

Here, $(\boldsymbol{\Sigma}, \boldsymbol{U}, W) = m_i$.

Then, the market clearing in the financial market m_i implies

$$
\begin{aligned}
& \left[\boldsymbol{I} + \frac{1}{\rho^2} \left(\boldsymbol{\Sigma} \boldsymbol{U} \right)^{-1} \right]^{-1} W + \rho \boldsymbol{\Sigma} F - \rho r \boldsymbol{\Sigma} P = W + Z \\
\Rightarrow \quad & \rho \boldsymbol{\Sigma} F = \left[\boldsymbol{I} - \left(\boldsymbol{I} + \frac{1}{\rho^2} \left(\boldsymbol{\Sigma} \boldsymbol{U} \right)^{-1} \right)^{-1} \right] W + \rho r \boldsymbol{\Sigma} P + Z \\
\Rightarrow \quad & F = \frac{1}{\rho} \left[\boldsymbol{\Sigma}^{-1} - \left(\boldsymbol{\Sigma} + \frac{1}{\rho^2} \boldsymbol{U}^{-1} \right)^{-1} \right] W + rP + \frac{1}{\rho} \boldsymbol{\Sigma}^{-1} Z.
\end{aligned}
$$

Hence, conditional on the price, investor i's belief over F is

$$
F | P \sim \mathcal{N} \left(\frac{1}{\rho} \left[\boldsymbol{\Sigma}^{-1} - \left(\boldsymbol{\Sigma} + \frac{1}{\rho^2} \boldsymbol{U}^{-1} \right)^{-1} \right] W + rP, \frac{1}{\rho^2} \boldsymbol{\Sigma}^{-1} \boldsymbol{U}^{-1} \boldsymbol{\Sigma}^{-1} \right).
$$

Then, investor i's optimal portfolio in the financial market m_i is

$$
\begin{aligned}
&\rho\mathbb{V}(F|P,S_i)^{-1}\left(\mathbb{E}(F|P,S_i)-rP\right) \\
=&\rho\left[\rho^2\boldsymbol{\Sigma U \Sigma}+\Omega_i\right]\times \\
&\quad\left\{\left[\rho^2\boldsymbol{\Sigma U \Sigma}+\boldsymbol{\Omega}_i\right]^{-1}\left[\rho^2\boldsymbol{\Sigma U \Sigma}\left(\frac{1}{\rho}\left[\boldsymbol{\Sigma}^{-1}-\left(\boldsymbol{\Sigma}+\frac{1}{\rho^2}\boldsymbol{U}^{-1}\right)^{-1}\right]W+rP\right)+\Omega_i S_i\right]-rP\right\} \\
=&\rho^2\boldsymbol{\Sigma U \Sigma}\left[\boldsymbol{\Sigma}^{-1}-\left(\boldsymbol{\Sigma}+\frac{1}{\rho^2}\boldsymbol{U}^{-1}\right)^{-1}\right]W+\rho\boldsymbol{\Omega}_i\left(S_i-rP\right) \\
=&\left[\rho^2\boldsymbol{\Sigma U}-\rho^2\boldsymbol{\Sigma U \Sigma}\left(\boldsymbol{\Sigma}^{-1}-\boldsymbol{\Sigma}^{-1}\left(\rho^2\boldsymbol{\Sigma U}+\boldsymbol{I}\right)^{-1}\right)\right]W+\rho\boldsymbol{\Omega}_i\left(S_i-rP\right) \\
=&\left[\boldsymbol{I}+\frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1}W+\rho\boldsymbol{\Omega}_i\left(S_i-rP\right) \\
=&X+D_i^*.
\end{aligned}
\tag{45}
$$

That is, given $d_j^*=1$ and $D_j^*=\rho\boldsymbol{\Omega}_j\left(S_j-rP\right)$ for all $j\in[0,1]\setminus\{i\}$, investor i's optimal portfolio in any $m_i\in\mathcal{M}_i$ is $(d_i^*,D_i^*)=(1,\rho\boldsymbol{\Omega}_i\left(S_i-rP\right))$.

Note that (d_i^*,D_i^*) is independent of m_i, so we have

$$
\begin{aligned}
&\inf_{m_i\in\mathcal{M}_i}\max_{(d_i,D_i)}U\left(m,(d_i,D_i)\right) \\
=&\inf_{m_i\in\mathcal{M}_i}U\left(m,(d_i^*,D_i^*)\right) \\
\leq&\max_{(d_i,D_i)}\inf_{m_i\in\mathcal{M}_i}U\left(m,(d_i,D_i)\right).
\end{aligned}
$$

Since generally

$$
\inf_{m_i\in\mathcal{M}_i}\max_{(d_i,D_i)}U\left(m,(d_i,D_i)\right)\geq\max_{(d_i,D_i)}\inf_{m_i\in\mathcal{M}_i}U\left(m,(d_i,D_i)\right),
$$

we have

$$\inf_{m_i \in \mathcal{M}_i} \max_{(d_i, D_i)} U\left(m, (d_i, D_i)\right) = \max_{(d_i, D_i)} \inf_{m_i \in \mathcal{M}_i} U\left(m, (d_i, D_i)\right).$$

Hence, (d_i^*, D_i^*) is a solution to $\max_{(d_i, D_i)} \inf_{m_i \in \mathcal{M}_i} U\left(m, (d_i, D_i)\right)$, implying that investor i does not have incentives to deviate from the prescribed investment strategy.

Q.E.D.

Proof of Proposition 6:

From equation (26), we can calculate the difference between individual assets' expected rates of return and the riskfree rate as

$$\text{diag}(P)^{-1}\mathbb{E}(F|P) - r\mathbb{1} = \text{diag}(P)^{-1}A_V, \tag{46}$$

where A_V is characterized in equation (55), and the expectation is taken conditional on the asset prices.

Since the pricing portfolio is *VWMP*, we calculate the market capitalization weights for individual assets as

$$\omega_V = \frac{\text{diag}(P)W}{P'W}. \tag{47}$$

Then, the variance of *VWMP* is

$$Var(R_V|P) = \left(\frac{1}{P'W}\right)^2 W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W, \tag{48}$$

and the covariances between individual assets and *VWMP* are

$$Cov(R, R_V|P) = \frac{1}{P'W}\text{diag}(P)^{-1}\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V. \tag{49}$$

Here, $\boldsymbol{C}_V$ is characterized in equation (57).

Therefore, when the pricing portfolio is *VWMP*, the assets' betas are

$$\beta_V = \frac{Cov(R, R_V)}{Var(R_V)} = P'W\frac{\text{diag}(P)^{-1}\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}, \tag{50}$$

and their alphas are calculated as in equation (30)

$$\alpha_V = \text{diag}(P)^{-1}\left[\boldsymbol{I} - \frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}\right]A_V.$$

Because $\frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W} \neq \boldsymbol{I}$, $\alpha_V = 0$ if and only if A_V is an eigenvector of $\frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}$ with the associated eigenvalue 1. Therefore, A_V should not be a function of $\boldsymbol{Q}$, since $\boldsymbol{I} - \frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}$ is not a function of $\boldsymbol{Q}$. However, it follows from Proposition 9 that $A_V = \frac{1}{\rho}\left[\rho^2\boldsymbol{\Sigma}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\right]^{-1}W$, and so A_V does depend on $\boldsymbol{Q}$. Therefore, A_V is not an eigenvector of $\frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}$ with the associated eigenvalue 1. Hence, $\alpha_V \neq 0$.

Q.E.D.

Proof of Proposition 7:

The proof is similar to that of Proposition 5. Consider the strategy profile that $(d_i^*, D_i^*) = (1, \rho\boldsymbol{\Omega}_i(S_i - rP))$ for all $i \in [0, 1]$. We argue that no investor wants to make a unilateral deviation.

Given all other investors' investment strategy, in a model in investor i's subjective belief $m_i = (\boldsymbol{\Sigma}, \boldsymbol{U}, W, \overline{F}, \boldsymbol{V}) \in \mathcal{M}_i$, the market clearing condition implies that

$$F = \frac{1}{\rho}\Sigma^{-1}\left[\boldsymbol{I} - \left(\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma U})^{-1}\right)^{-1}\right] W - \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{I} + \rho^2 \boldsymbol{\Sigma U}\right]^{-1} V(\overline{F} - rP) + rP + \frac{1}{\rho}\boldsymbol{\Sigma}^{-1} Z.$$

Hence, $F|P$ is normally distributed with mean

$$\frac{1}{\rho}\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{I} - \left(\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma U})^{-1}\right)^{-1}\right] W - \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{I} + \rho^2 \boldsymbol{\Sigma U}\right]^{-1} \boldsymbol{V}(\overline{F} - rP) + rP$$

and the precision

$$\rho^2 \boldsymbol{\Sigma U \Sigma}.$$

Therefore, conditional on the prior, and the asset prices, and her own private signals (in model m_i), investor i's optimal portfolio is

$$\begin{aligned}
&\left[\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma U})^{-1}\right]^{-1} W + \rho\left[\boldsymbol{I} + \rho^2 \boldsymbol{\Sigma U}\right]^{-1} V\left(\overline{F} - rP\right) + \rho \boldsymbol{\Omega}_i \left(S_i - rP\right) \\
=& Y + D_i^*,
\end{aligned}$$

implying that $(d_i^*, D_i^*) = (1, \rho \boldsymbol{\Omega}_i (S_i - rP))$ is optimal in the model m_i.

Furthermore, (d_i^*, D_i^*) is independent of m_i. Hence, by the exactly same argument at the end of the proof of Proposition 5, we show that (d_i^*, D_i^*) is the solution to the investor i's maxmin problem, provided that all other investors trade as prescribed.

Q.E.D.

Proof of Proposition 8:

The proof is also similar to that of Proposition 5. For a given model $m_i = (\overline{\mathbf{\Sigma}}, \boldsymbol{U}, W, \overline{\rho})$, when all other investors trade as prescribed, investor i believes that the market clearing condition is

$$\int_0^1 \left(\rho_j \left[\overline{\rho}\boldsymbol{I} + \left(\overline{\mathbf{\Sigma}}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho_j \mathbf{\Omega}_j \left(S_j - rP\right)\right) \mathrm{d}j = W + Z,$$

which implies that the pricing function is

$$F = \overline{\mathbf{\Sigma}}^{-1} \left[I - \overline{\rho} \left[\overline{\rho}\boldsymbol{I} + (\overline{\mathbf{\Sigma}}\boldsymbol{U})^{-1}\right]^{-1}\right] W + rp + \overline{\mathbf{\Sigma}}^{-1} Z.$$

Hence, the asset payoffs conditional on asset prices have the distribution

$$F|P \sim \mathcal{N}\left(\overline{\mathbf{\Sigma}}^{-1} \left[I - \overline{\rho} \left[\overline{\rho}\boldsymbol{I} + (\overline{\mathbf{\Sigma}}\boldsymbol{U})^{-1}\right]^{-1}\right] W + rp, \overline{\mathbf{\Sigma}}^{-1} \boldsymbol{U}^{-1} \overline{\mathbf{\Sigma}}^{-1}\right).$$

Then, investor i's optimal portfolio choice in the financial market m_i is

$$\begin{aligned} & \rho_i \mathbb{V}(F|P, S_i)^{-1} \left[\mathbb{E}(F|P, S_i) - rp\right] \\ = \; & \rho_i \left[\overline{\rho}\boldsymbol{I} + \left(\overline{\mathbf{\Sigma}}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho_i \mathbf{\Omega}_i \left(S_i - rP\right) \\ = \; & \rho_i \overline{X} + \rho_i \mathbf{\Omega}_i \left(S_i - rP\right). \end{aligned}$$

That is, investor i's optimal investment strategy in the perceived financial market m_i is $(d_i^*, D_i^*) = (\rho_i, \rho_i \mathbf{\Omega}_i (S_i - rP))$. Because such an investment strategy is independent of m_i, by the same argument at the end of the proof of Proposition 5, we can show that investor i's optimal investment strategy is (d_i^*, D_i^*).

Therefore, given investor $j \in [0,1] \setminus \{i\}$ employs the investment strategy $\left(d_j^*, D_j^*\right) = \left(\rho_j, \rho_j \mathbf{\Omega}_j \left(S_j - rP\right)\right)$, investor i will not deviate, proving that the strategy under consideration is an equilibrium.

Q.E.D.

B. Equilibrium Asset Holdings with a *VWMP* Fund

In this appendix, we discuss the investors' equilibrium asset holdings and the asset pricing implications when the fund is offering the investors with the value-weighted market portfolio (*VWMP*); formally, we assume that the index fund offers $X_V = W$. We show that ambiguity averse investors are not willing to hold a fund offering *VWMP* as the passive (noninformational) index component of their portfolios. This establishes a benchmark for our analysis when the passive fund is offering *RAMP*.

To keep such a benchmark model tractable, we assume that all financial market parameters, except the precisions of random supply shocks to risky assets, are common knowledge among investors. Therefore, the endowed portfolio W and the average precision of investors' private signals are commonly known.[17] We further assume that all assets are independent. That is, $\mathbf{\Omega}_i$ is diagonal for all $i \in [0,1]$. Then, investor i is an informed investor of asset n if and only if the n^{th} diagonal entry of $\mathbf{\Omega}_i$ is strictly positive. Let λ_n be the measure of informed investors of asset n; we assume that $\lambda_n \in (0,1)$, for all $n \in \mathcal{Q}$. Let $\text{diag}(\lambda)$ be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being λ_n.

For simplicity, we assume that the private signals of all informed investors of asset n have the same precision $\kappa_n > 0$. Let $\mathbf{\Omega}$ be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being κ_n. Letting $\mathbf{\Sigma}$ be the matrix of the average precision of private signals, we have

$$\mathbf{\Sigma} = \int_0^1 \mathbf{\Omega}_i \mathrm{d}i = \mathbf{\Omega}\text{diag}(\lambda) \tag{51}$$

[17] By assuming that investors commonly know some of the financial market parameters but are ambiguous about some other parameters, we show that the effect of the *VWMP* is independent of the financial market parameters investors are ambiguous about.

Because assets are independent, $\boldsymbol{U}$ is also diagonal. We focus on the case that all model parameters are commonly known among investors except $\boldsymbol{U}$. We assume that for each asset n, a subset of uninformed investors do not know u_n. We say that such a group of investors are subject to *model uncertainty* (or are *ambiguous*) about asset n. Any investor i who is ambiguous about asset n will have her own subjective prior belief about u_n with the support $\left(0, \bar{u}_n^i\right)$, where $\bar{u}_n^i > 0$. So we allow different investors who are ambiguous about a particular asset n to have different supports of their beliefs about u_n. Let $\mathcal{U}_i$ be the set of all possible subjective beliefs of investor i about $\boldsymbol{U}$, let $\boldsymbol{U}_i$ be a typical element in $\mathcal{U}_i$, and let $\underline{\boldsymbol{U}}_i$ be the lower bound of $\mathcal{U}_i$.

Let the measure of the group of investors who are ambiguous about n be $1 - q_n \in (0, 1 - \lambda_n)$. Let $\boldsymbol{Q}$ be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being q_n. For simplicity, we assume that an investor who is ambiguous about asset n is also uninformed about asset n. However, an investor who is uninformed about asset n may know u_n and so is not ambiguous about asset n.

Each investor i's decision problem is then

$$\max_{d_i, D_i} \inf_{\boldsymbol{U}_i \in \mathcal{U}_i} \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{52}$$

Investor i is risk averse, so she only holds a non-zero position of asset n, if her subjective belief of asset n's payoff has a finite variance, conditional on her information. When investor i is uninformed about asset n, however, she has neither prior information nor private information about the payoff of asset n. Hence, she estimates the payoff based only on the price, which partially aggregates informed investors' private information. Since the precision of asset n's random supply, u_n, is strictly positive (no matter how small it is), if investor i knows u_n, her belief of asset n's payoff has a finite variance and, therefore, she will hold a non-zero position of asset n.

On the other hand, if investor i is subject to model uncertainty about asset n, she does not know the precision of asset n's random supply. By assumption, investor i's subjective prior about u_n has the support $\left(0, \overline{u}_n^i\right)$. Since all random variables in the model are normally distributed, observing the asset price does not change the support of investor i's belief about u_n, although investor i may extract some information about u_n from asset n's price. Hence, the worst-case scenario is independent of asset n's price. Specifically, when the precision of the random supply is arbitrarily close to zero, price becomes almost uninformative. So as investor i considers the worst-case scenario in making the investment decision, she focuses on the possibility that the true u_n is very close to 0. For any non-zero position of asset n, as the price becomes almost uninformative, the payoff variance conditional upon investor i's information diverges to infinity. So holding a non-zero position is extremely risky in the worst-case scenario. To avoid this risk, investor i optimally chooses a zero position. Lemma 1 below summarizes the argument above.

Lemma 1. *An investor i who is ambiguous about asset n optimally holds a zero position in it.*

Proof of Lemma 1:

Because investor i is ambiguous about asset n, by assumption, she does not have private signal about asset n's payoff; that is, $\kappa_i = 0$. Hence, investor i's only information about the distribution of asset n's payoff is its price, which may partially aggregate informed investors' private signals. Suppose the uninformed investors' aggregate demand for asset n is $(1 - \lambda_n)D(p_n)$. Since uninformed investors do not observe u_n, $D(p_n)$ is not a function of u_n.

Given any P and any $u_n \in \left(0, \overline{u}_n^i\right)$, we derive investor i's expected utility conditional on P_n as follows. Suppose asset n's pricing function in a linear equilibrium is

$$f_n = a + bp_n + cz_n,$$

where a, b, and c are undetermined parameters. Since informed investors know u_n, they can extract information from the price without any ambiguity. Therefore, any informed investor j's demand is

$$D_j = \rho \left[\kappa_n s_j + \frac{u_n}{c^2} a + \frac{u_n}{c^2} (b - r) p_n - r\kappa_n p_n \right].$$

Then, the informed investors' aggregate demand will be

$$\lambda_n \rho \left[\kappa_n f_n + \frac{u_n}{c^2} a + \frac{u_n}{c^2} (b - r) p_n - r\kappa_n p_n \right].$$

Then, the market clearing condition implies that

$$\lambda_n \rho \left[\kappa_n f_n + \frac{u_n}{c^2} a + \frac{u_n}{c^2} (b - r) p_n - r\kappa_n p_n \right] + (1 - \lambda_n) D(P_n) = w_n + z_n.$$

Matching the coefficient of the market clearing condition and the pricing function, we have

$$\begin{aligned} a &= \frac{w_n}{\lambda_n \kappa_n \rho} - \frac{u_n}{c^2 \kappa_n} a \\ bp_n &= -\frac{(1 - \lambda_n) D(p_n)}{\lambda_n \kappa_n \rho} - \frac{u_n}{c^2 \kappa_n} (b - r) p_n + rp_n \\ c &= \frac{1}{\lambda_n \kappa_n \rho} \end{aligned}$$

Therefore, for any given $u_n \in \left(0, \overline{u}_n^i\right)$, conditional on the price P_n, $|\mathbb{E}(f_n - rp_n | p_n)| < +\infty$.

On the other hand, the variance of asset n's payoff conditional on p_n is

$$\mathbb{V}\left(f_n|p_n\right) = c^2 u_n^{-1},$$

which diverges to $+\infty$ as u_n goes to 0. Hence, any non-zero position D_i of asset n brings investor i a utility

$$-\exp\left(-\frac{1}{\rho}w_i r p_n\right)\exp\left[-\frac{1}{\rho}D_i\mathbb{E}\left(f_n - rp_n|p_n\right) + \frac{D_i^2}{2\rho^2}\mathbb{V}\left(f_n|p_n\right)\right], \tag{53}$$

which goes to $-\infty$ as u_n goes to 0. Therefore, if investor i is ambiguous about asset n, investor i will hold a zero position of asset n.

Q.E.D.

It directly follows Lemma 1 that if an investor is ambiguous about an set of risky assets, then she will not hold the *VWMP* because it contains strictly positive positions in each risky assets.

We now analyze the investors' equilibrium asset holdings. The model is similar to the rational expectations equilibrium model with multiple risky assets Admati (1985). The key difference is that for each asset n, there are $1 - q_n$ measure investors who will hold a zero position (by Lemma 1). Proposition 9 below characterizes a linear rational expectations equilibrium.

Proposition 9. *In the model where the passive fund offers VWMP (formally, $X_V = W$), there exists a linear equilibrium with the pricing function*

$$P = \boldsymbol{B}_V^{-1}\left[F - A_V - \boldsymbol{C}_V Z\right], \tag{54}$$

where

$$A_V = \frac{1}{\rho}\left[\rho^2(\boldsymbol{\Sigma QU\Sigma}) + \boldsymbol{\Sigma}\right]^{-1} W \quad (55)$$

$$\boldsymbol{B}_V = r\boldsymbol{I} \quad (56)$$

$$\boldsymbol{C}_V = \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}. \quad (57)$$

Any investor i's effective risky asset holding is

$$d_i X + D_i = \lim_{\boldsymbol{U}_i \to \underline{\boldsymbol{U}}_i} \left[\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma U}_i)^{-1}\right]^{-1} W + \rho \boldsymbol{\Omega}_i (S_i - rP) \quad (58)$$

Proof of Proposition 9:

We assume that the pricing function can be written as

$$F = A_V + B_V P + C_V Z,$$

where B_V is nonsingular. Then, conditional on the asset prices P, an investor i's updated belief about asset payoffs is

$$F|P \sim \mathcal{N}\left(A_V + \boldsymbol{B}_V P, \boldsymbol{C}_V' \boldsymbol{U}_i^{-1} \boldsymbol{C}_V\right),$$

where $\boldsymbol{U}_i \in \mathcal{U}_i$

Then, each investor i's optimal asset holdings are

$$d_i X + D_i = \rho\left\{(\boldsymbol{C}_V \boldsymbol{U}_i^{-1} \boldsymbol{C}_V')^{-1}(\boldsymbol{B}_V - r\boldsymbol{I}) - r\boldsymbol{\Omega}_i\right\} P + \rho \boldsymbol{\Omega}_i S_i + \rho[(\boldsymbol{C U}^{-1} \boldsymbol{C}')^{-1} A_V]. \quad (59)$$

It follows from Lemma 1 that if investor i is ambiguous about asset n, her holding of asset n is zero. Then, aggregating all investors' effective asset holdings and applying the market clearing condition yield

$$\begin{aligned} D &= \rho\left[\left(\boldsymbol{C}_V'\right)^{-1}\boldsymbol{QU}\left(\boldsymbol{C}_V\right)^{-1}A_V\right]+\rho\boldsymbol{\Sigma}F+\rho\left[\left(\boldsymbol{C}_V'\right)^{-1}\boldsymbol{QU}\left(\boldsymbol{C}_V\right)^{-1}\left(\boldsymbol{B}_V-r\boldsymbol{I}-r\boldsymbol{\Sigma}\right)\right]P \\ &= W+Z. \end{aligned}$$

Therefore, by matching coefficients, we have

$$\begin{aligned} \boldsymbol{C}_V &= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1} \\ \boldsymbol{B}_V &= r\boldsymbol{I} \\ A_V &= \frac{1}{\rho}\left[\rho^2\boldsymbol{\Sigma}\boldsymbol{QU}\boldsymbol{\Sigma}+\boldsymbol{\Sigma}\right]^{-1}W. \end{aligned}$$

Substituting these parameters into the pricing function and the individual investor's asset holding function, we get Proposition 9.

Q.E.D.

Equations (54) and (55) show that for each risky asset n, the measure of investors who know the precision of its random supply affects its equilibrium price. In particular, $\boldsymbol{QU}$ is the matrix of the average precisions of asset random supplies in the investors' subjective "worst-case scenarios," which positively affect the asset prices. Hence, *ceteris paribus*, if $q_k > q_n$, the equilibrium price of asset k is greater than that of asset n. Intuitively, when $q_k > q_n$, on average the subjective worst case for k is not as bad as for n, so the demand function for asset k is higher than the demand function for asset n. So when both assets have the same supply, asset k's price is higher than that of asset n.

Equation (58) characterizes investor i's effective risky asset holdings in equilibrium. For each asset n, if investor i knows the precision of its random supply, she will hold a position based on the equilibrium price. Formally, in such a case, the n^{th} diagonal entry of $\underline{\boldsymbol{U}}_i$ is $u_n > 0$; hence, the first term in equation (58) is positive. Furthermore, if such an investor receives a private signal about asset n's payoff, the n^{th} diagonal entry of $\boldsymbol{\Omega}_i$ is $\kappa_n > 0$, and so the second term is also positive.

At the other extreme, if investor i is ambiguous about asset n, both the n^{th} diagonal entry of $\underline{\boldsymbol{U}}_i$ and the n^{th} diagonal entry of $\boldsymbol{\Omega}_i$ are zero, implying that investor i holds a zero position of asset n. Therefore, even with a passive fund offering *VWMP*, ambiguity averse investors will not participate in some assets' markets. As a result, *VWMP* is not effective in encouraging ambiguity averse investors to hold better-diversified portfolios.

Interestingly, it follows from equation (58) that the upper bound of $\mathcal{U}_i$, denoted by $\bar{\boldsymbol{U}}_i$, does not affect the equilibrium outcomes when the passive fund commits to offering the *VWMP*. This is because investor i will perceive higher risks of holding the *VWMP* as $\boldsymbol{U}_i$ becomes smaller. Hence, because of the investor's max-min utility, only the lower bound of her belief support, $\underline{\boldsymbol{U}}_i$, matters for her investment decision. Therefore, if we define the degree of ambiguity of an investor i by $|\bar{\boldsymbol{U}}_i - \underline{\boldsymbol{U}}_i|$, then the willingness of investor i to invest in the passive fund that offers the *VWMP* varies with her degree of ambiguity only when $\underline{\boldsymbol{U}}_i$ changes.

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C. Online Appendix (not for Publication)

A. Single-Factor Model with *RAMP* as Common Factor

An interesting asset pricing implication of the model is that asset risk premia satisfy the CAPM with *RAMP* as the pricing portfolio. This implication holds even if investors possess asymmetric information and are subject to ambiguity aversion, as formally stated in Corollary 1. Such an asset pricing implication suggests that *RAMP* performs better than *VWMP* in a single-factor pricing model. In this appendix, we present supporting empirical evidence.[1]

As discussed in Section IV, *RAMP* includes smaller positions of more volatile stocks and so can be viewed as a low-volatility portfolio. Therefore, it is natural to use the low-volatility ETF that is traded in the market as a proxy for *RAMP*. One candidate is the Invesco S&P 500® Low Volatility ETF (ticker: SPLV), which tracks the S&P 500® Low Volatility Index.[2] The index measures performance of the 100 least volatile stocks in the S&P 500, with weights proportional to the reciprocal of historical volatility (among the 100 least volatile stocks in the S&P 500).

Since the index was launched in 2011, the sample period is from June 2011 to December 2020. Table 1 below shows basic statistics of *VWMP*, and the Invesco S&P 500® Low Volatility ETF. We see that the low volatility index has a lower average return and a lower level of volatility.

	VWMP	*SPLV*
Average return (%)	1.18	0.97
Standard deviation (%)	4.18	3.17

Table 1: Return and volatility of *VWMP* and *SPLV* from June 2011 **to December** 2020

Table 2 shows estimates of a single-factor model with *VWMP* and the Invesco S&P 500® Low Volatility ETF as common factors. We see that the low volatility ETF performs better than *VWMP* for pricing assets in cross section. This is consistent with the idea that assets should approximately be priced by a proxy for *RAMP* rather than *VWMP*. With *VWMP* being the pricing portfolio, the average absolute value of alphas is 0.409% and is statistically significant (with the t-statistic being 1.66). When we replace *VWMP* with the low volatility ETF, the average absolute value of alphas decreases to 0.252% and becomes insignificant (with the t-statistic being 0.65).

A weakness of the low volatility ETF as pricing portfolio in the test in Table 2 is that the R^2 is low. This may be due to the short sample period and the fact that the index that the low volatility ETF tracks includes 100 stocks only. We therefore construct a volatility-weighted market portfolio

[1] To perform the empirical tests, we follow Fama and French (1993) to construct testing portfolios. Specifically, we use the 5 × 5 portfolios based on size and book-to-market ratio as the test portfolios.

[2] See "S&P 500 Low Volatility Index: Five Decades of History," by the research department of S&P Global for the comparison between S&P 500® Low Volatility Index and S&P 500 Index.

	VWMP	*SPLV*
$\lvert\alpha\rvert$	0.409	0.252
t	1.66	0.65
R^2	0.81	0.46

Table 2: A Single-Factor Model with *VWMP* and *SPLV* as common factors from June 2011 **to December** 2020 **(estimated α in percentage)**

as a proxy for *RAMP*, which weights the returns of assets in the *VWMP* by the reciprocal of their standard deviations (without limiting positive weights to the 100 lowest volatility stocks). This allows us to run tests over a longer period than using the low-volatility ETF. Specifically, we construct a portfolio *MKTSD* as

$$MKTSD_t = \frac{\sigma_{it}^{-1}}{\sum \sigma_{it}^{-1}} Ret_{it}, \tag{60}$$

where Ret_{it} is the monthly return of stock i in month t, and σ_{it} is the standard deviation estimated using the three-year rolling window of monthly returns.

We then test the single-factor model using *VWMP* and *MKTSD* as common factors over the sample period from December 1963 to December 2020. The estimations reported in Table 3 show that *MKTSD* outperforms *VWMP* in all three metrics (i.e., $|\alpha|$, t, and R^2).

Table 3: A Single-Factor Model with *VWMP* and *MKTSD* as common factors from December 1963 **to December** 2020 **(estimated α in percentage)**

	VWMP	*MKTSD*
$\lvert\alpha\rvert$	0.199	0.153
t	1.72	1.45
R^2	0.748	0.784

B. Relation between *RAMP* and Low Volatility Indexes

In Online Appendix A, we use the S&P 500® Low Volatility ETF (ticker: SPLV), which tracks the S&P 500® Low Volatility Index, as a proxy for *RAMP* to conduct a preliminary empirical test. Our results indicate that the low-volatility index outperforms *VWMP* in pricing assets. However, as discussed in Section C, this empirical test has caveats. In particular, the S&P 500® Low Volatility Index differs from *RAMP* in key aspects, so even if a tradable ETF closely tracks it, it may not effectively reduce ambiguity premia.

However, other tradable ETFs that track alternative low volatility indexes — beyond the S&P 500® Low Volatility Index — or a combination of such ETFs may provide a better approximation of *RAMP*.[3] Identifying such indexes would require a substantial empirical study, including the estimation of all relevant market parameters. Nonetheless, it is valuable to explore, from a theoretical perspective, how a low-volatility index should be designed to approximate *RAMP* and how its successful implementation could affect pricing errors. In this appendix, we address this question through numerical studies.

We first consider a setting in which assets are independent — that is, both investors' private information precision matrices and the random supply shock precision matrix are diagonal. In this case, we follow the common approach of constructing a low-volatility portfolio by assigning positive weights to the K lowest-volatility assets (out of a total of N assets) while setting the weights of all other assets to zero. However, unlike the S&P 500® Low Volatility Index, we determine the weighting function based on *RAMP*. As a result, the low-volatility portfolio we construct is effectively a component of *RAMP*, ensuring full participation in the markets for assets included in the portfolio.[4] Consequently, this low-volatility portfolio effectively reduces ambiguity premia, leading to smaller pricing errors.

Building on this theoretical foundation, we perform a numerical simulation to compare asset pricing errors under three scenarios that differ based on which fund is offered to investors: *RAMP*, our low-volatility fund, and *VWMP*. In the simulation, there are 500 independent assets. We randomly draw all the parameters so that all the precision matrices are diagonal and have full rank. We then pick the 300 lowest-volatility assets to construct a low volatility index.[5]

Figure 1 summarizes the results. We find that the pricing error is decreasing as we vary the available index fund to investors from *VWMP* to the low volatility fund and then to *RAMP*. This shows that a well-designed low volatility fund can serve as a stepping stone to the implementation of *RAMP*. The difference in pricing errors shrinks as the fraction of rational investors increases, as pricing errors derive from ambiguity premia.

We then consider the case where assets are correlated (i.e., when the relevant parameter matrices are not diagonal). First, as shown in Van Nieuwerburgh and Veldkamp (2009), the returns of correlated assets can be decomposed into a finite set of independent factors. Therefore, without

[3] Unlike the S&P 500® Low Volatility Index, which employs a simple inverse volatility weighting method (assigning weights proportional to the reciprocals of volatilities), some low-volatility indexes use alternative weighting approaches, such as the Equal Risk Contribution method. These indexes are generally less transparent than the S&P 500® Low Volatility Index, as they disclose only broad principles of their construction methods without revealing many details.

[4] This follows from the properties of mean-variance utility: When assets are independent, an investor's demand for one asset is independent of her demand for another. Then, using a similar argument to the proof of Proposition 5, we show that all investors will participate in the markets for assets in the low-volatility portfolio, while ambiguity averse investors will refrain from trading assets outside the portfolio.

[5] Intuitively, the more assets we include in the low-volatility portfolio, the lower the ambiguity premia and thus the smaller the pricing errors with the low-volatility portfolio.

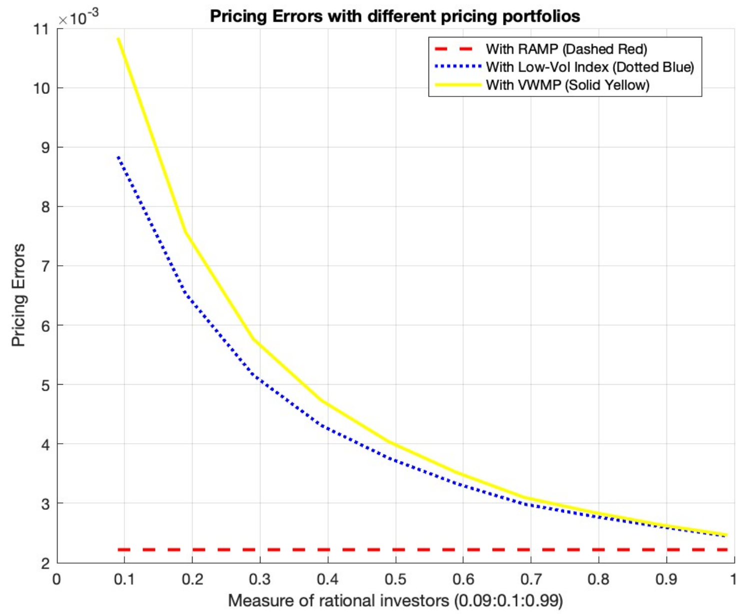

Figure 1: Simulated pricing errors when assets are uncorrelated.

loss of generality, we can focus on financial markets characterized by independent factors, where investors may be ambiguity averse about the parameters of the distributions of these factors. In such a setting, a low-volatility portfolio constructed based on independent factors can serve as a good approximation of *RAMP*, thereby encouraging market participation.

However, for portfolios constructed from individual stocks instead of from factors, it is necessary to take into account security correlations to match *RAMP*. We conduct a numerical study under a setting where assets are independent, except for their random supply shocks. Specifically, we assume that the precision matrix of these shocks, $\boldsymbol{U}$, is non-diagonal but has a block structure:

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{U}_L & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_H \end{pmatrix},$$

where both the matrices $\boldsymbol{U}_L$ (with the dimension $K \times K$) and $\boldsymbol{U}_H$ are non-diagonal. We further assume that assets corresponding to $\boldsymbol{U}_L$ have relatively low conditional volatility (given asset prices). Based on this, we construct a low-volatility portfolio following the *RAMP* framework but use the K assets associated with $\boldsymbol{U}_L$.

Since this low-volatility portfolio, when offered to investors publicly, enhances market participation and reduces ambiguity premia, it will lower pricing errors. This supports its role as a potential stepping stone toward implementing *RAMP*. The simulation results, summarized in Figure 2, confirm this finding.

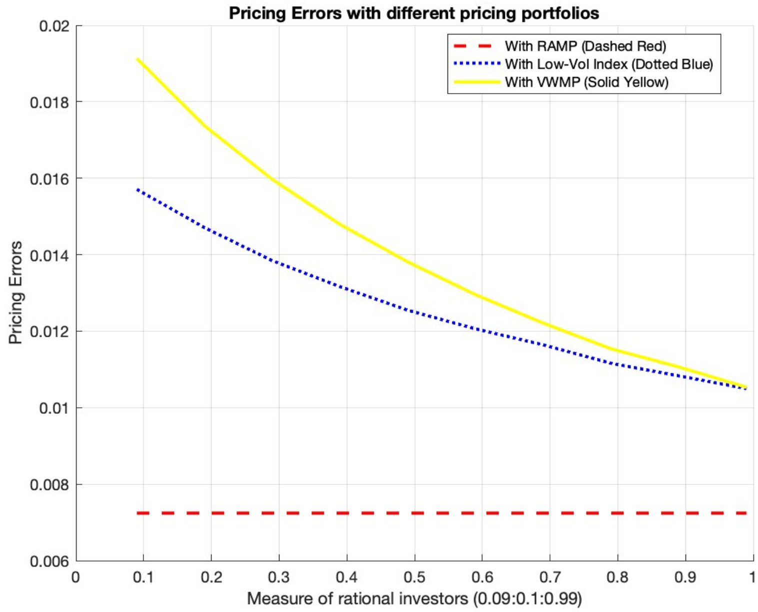

Figure 2: Simulated pricing errors when assets are uncorrelated.

C. Implementation of the *RAMP* Fund

We have shown that our new index, *RAMP*, can encourage all investors, including ambiguity averse ones, to engage in index investing, and thus in equilibrium investor risky asset holdings are exactly same as those in the economy without model uncertainty. Therefore, in theory, any possible inefficiency due to investors' model uncertainty can be corrected by an index fund that commits to offering *RAMP*. However, in practice who would we expect to be well-positioned to offer a *RAMP* fund? In this section, we discuss this question.

Investors have long been benefiting from specialization in the money management industry. Some agents, such as index committees or companies, have professional knowledge to estimate financial market parameters, while other agents, such as mutual funds or hedge funds, focus on trading based on information about specific companies' fundamentals. Equation (13) shows that constructing *RAMP* requires knowledge about the relevant parameter values that characterize the financial market. This kind of knowledge does not include any information related to specific companies' fundamentals, which is the focus of active funds. Hence, a *RAMP* index could be constructed and maintained by an index committee or company, just as is the case with most popular indexes in the current stock market.[6]

Once the *RAMP* index is available, the index committee or company can license index funds

[6]For example, the S&P 500 index is maintained by the U.S. Index Committee whose members are full-time professional S&P Dow Jones Indices' staffs. The committee meets on a monthly basis to review and update the index composition. Similarly, Russell 3000 Index is constructed and maintained by FTSE Russell Group.

and ETFs to replicate and offer it to investors. There may be agency problems associated with these fund providers, a topic that we do not focus on in this paper. However, as Berk (2005) points out, there are implicit contracts between funds and their investors. Index funds' future investment flows depend on how well the funds track the index in the current period. This provides funds with incentives to track the index closely. Huang, Li, and Weng (2020) formalize such implicit contracts as fund reputations, which are summarized by Morningstar ratings. Therefore, we regard reputation as a tool by which index funds commit to providing *RAMP* to investors.

The aforementioned approach of implementing a *RAMP* fund requires an agent, such as an index committee or an index company, to estimate all relevant parameters of the financial market. Fast-developing information processing technologies and "big data" are likely to further improve the feasibility of a fund estimating all relevant financial market parameters for the formation of *RAMP*. *Implementing RAMP* also requires that funds be long-lived and so care about their future flows, i.e., their reputation serves as a commitment tool. We suggest above that these two requirements are largely satisfied in current financial markets.

Furthermore, we next show that in the model, *RAMP* funds can be offered to investors even if no single agent (i.e., fund manager) knows all parameters about the whole capital market, and even if funds do not care about reputation.

Inspired by the fund-of-funds investment strategy and the fund disclosure requirements, we consider a mechanism design approach to implementing *RAMP* as a fund of funds. The main idea is to have agents (fund managers) who know parameters of a subset of assets build "sector funds." This is in the spirit of the recent rise in popularity of sector or thematic ETFs. In the model, a regulator requires funds to disclose their holdings, with penalties for failing to hold assets as the funds promise. Investors then construct *RAMP* using all sector funds.

Specifically, without loss of generality, we follow Van Nieuwerburgh and Veldkamp (2009) to assume that all traded assets are decomposed into Q independent risk factors. Then, the vector of factor payoffs is $F = (f_1, f_2, \ldots, f_Q)'$. Any investor i's private signal, $S_i = F + \epsilon_i$, is then about risk factor payoffs, and the precision matrix $\boldsymbol{\Omega}_i$ of the noise term ϵ_i is diagonal. We further assume that the precision matrix $\boldsymbol{U}$ of the random supply Z is also diagonal. We denote by H^j the j^{th} element of *RAMP* defined in equation (13), so $X = \left(H^1, H^2, \ldots, H^Q\right)'$.

Importantly, while there may not be one investor who knows all parameters that characterize the whole financial market, for each factor j, there is a positive measure of investors who know all the parameters about it. To keep the analysis simple, we assume that these investors do not possess any private signals about factor j payoff, nor do they know parameters of any other factors. That is, these investors specialize in factor j market and are called "factor-j-uninformed investors."

We consider a mechanism Γ that is publicly announced and enforced by a regulator. There are three dates. At date one, based on her knowledge about the financial market and her information about factor payoffs, each investor k chooses to offer a **passive fund** or not, and if she does, she specifies the factors she will cover and charges a management fee τ_k. Since investor k does not acquire new information, we assume that her marginal cost of offering a passive fund is zero. We denote by $\mathcal{A}_j$ the set of passive funds that cover factor j. At date two, investors trade all individual assets and all funds offered, as well as the riskfree asset. At date three, each fund discloses its asset holding. Let H_k^j be the shares of factor j included in one share of fund k for any $k \in \mathcal{A}_j$. Then, the regulator calculates the median shares of factor j included in one fund share among funds that cover factor j. Formally, define

$$\widehat{H}^j = \underset{k \in \mathcal{A}_j}{\text{Median}}\{H_k^j\}. \tag{61}$$

The regulator then imposes a fine of $\left|H_k^j - \hat{H}^j\right| B$ on fund k, where $B > 0$ is sufficiently large. The total fines collected in factor j will be distributed among funds whose one-share holdings of factor j are exactly $\hat{H}^j$, so that the regulator has a balanced budget.

It is straightforward that in the environment described above, there is no single agent (including the regulator) who knows all the parameters that characterize the whole financial market. Also, the mechanism Γ is designed for a static setting, so there are no future flows that can incentivize funds to choose the optimal portfolios for their investors. Nevertheless, we show that the equilibrium outcome in Proposition 5 is *Nash-implementable* by Γ. That is, as shown in Proposition 10, the mechanism Γ has a Nash equilibrium in which investors effectively hold assets exactly as they do in Proposition 5.[7]

Proposition 10. *The mechanism Γ has a Nash equilibrium in which*

1. *Any factor-j-uninformed investor k offers a sector fund, and one share of its holding is $H_k^j = H^j$.*

2. *No investor outside the group of factor-j-uninformed investors offers a passive fund that covers factor j.*

3. *The management fee charged by any sector passive fund k is $\tau_k = 0$.*

4. *An investor i holds one share at one and only one sector fund that covers factor j (for all j).*

5. *Any investor i has an extra holding $\rho\mathbf{\Omega}_i\left(S_i - rP\right)$.*

[7]Following the literature on mechanism design, we focus on whether the equilibrium outcome characterized in Proposition 5 is implementable or not by the mechanism Γ. That is, whether there is an equilibrium of Γ in which investors have exactly same holdings as in Proposition 5. There may be multiple equilibria of Γ, but the equilibrium multiplicity is not a focus in our paper, as it is not the focus in other mechanism design problems.

Proof of Proposition 10:

Given that $\mathcal{A}_j$ is just the group of factor-j-uninformed investors, the strategy profile described in Part 1 and Part 2 implies that $\widehat{H}^j = H^j$. Then, if any investor k offers $H_k^j \neq H^j$, the regulator imposes a fine $\left|H_k^j - H^j\right| B > 0$. Hence, such a deviation is not profitable. Therefore, no fund will deviate from the portfolio it offers. Part 3 is simply due to the competition among all sector funds that covers the same factor.

Part 4 and Part 5 imply that any investor's asset holding is exactly the same as in Proposition 5. Therefore, by the proof of Proposition 5, no investor wants to deviate. This completes the proof.

Q.E.D.

The first two parts of Proposition 10 follow from the designed transfers in Γ. Since all factor-j-uninformed investors know the parameters that characterize the market of factor j, they can hold H^j. On the other hand, in equilibrium, the median holding of factor j is $\widehat{H}^j = H^j$. Hence, any unilateral deviation by a fund k will lead to a positive fine. Therefore, only factor-j-uninformed investors offer sector funds that cover factor j, and since all sector funds that cover the same factor offer exactly the same portfolio, the fine to each of them is zero. Furthermore, since there is a continuum of funds that offer the same holding, their competition will drive the management fees down to zero, which is stated in part 3 of Proposition 10.

Part 4 is a fund-of-funds idea. All investors hold one share at one and only one sector fund that covers factor j (for all j), so their holdings through funds are $(H^1, H^2, \ldots, H^Q)'$, which is exactly *RAMP* according to the definition of H^j. Part 5 is about investors' own information-based portfolios: by the information separation theorem, investors do not need any information about the financial market to construct their own information-based portfolios. Part 4 and Part 5 together imply that in equilibrium, investors are effectively holding one share of *RAMP* and their own information-based portfolios, which are their optimal investment strategies in the equilibrium described in Proposition 5.

The two key assumptions underlying this implementation of *RAMP* with the mechanism Γ are indeed plausible. The first is that there is competition among sector funds. This is needed because when there are sufficiently many sector funds who cover the same factor, competition can drive down the management fee, and the regulator can calculate the median share of each factor to implement the transfer system. This is similar to a necessary condition of Nash implementation in mechanism design problems: there must be at least three agents. In actual financial market, significant factors such as industry factors and regional factors, are usually covered by multiple index funds and ETFs. Hence, the assumption of sector funds' competition is plausible.

The other assumption is the disclosure of fund holdings. Without such a requirement, the regulator cannot compare funds' holdings and implement the transfer system. In practice, index funds are required to make quarterly disclosure about their asset holdings. ETFs are even more transparent and disclose holdings in a daily basis.