Financing Innovation under Ambiguity

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Abstract

We develop a real options model in which an entrepreneur facing ambiguity makes optimal investment and financing decisions for an innovation project. We introduce jumps in innovation returns and model investors' aversion to ambiguity in both diffusion and jump risks. Debt accelerates investment by lowering the threshold and shortening expected waiting time, thereby increasing project value. This effect strengthens under greater ambiguity, offering a novel rationale for why debt—not equity—fosters innovation. Our results provide a coherent explanation for recent empirical findings on debt's role in innovation and contribute to the broader literature on investment under uncertainty.

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I. Introduction

Researchers have found that debt is essential for funding innovations across diverse contexts, including publicly traded companies (Mann, 2018) and startups (Robb and Robinson, 2014; Davis, Morse, and Wang, 2020). Furthermore, empirical studies have shown that debt fosters innovation and that improved debt financing accelerates innovation and cultivates innovation novelty for firms (Benfratello, Schiantarelli, and Sembenelli, 2008; Amore, Schneider, and Žaldokas, 2013; Chava, Oettl, Subramanian, and Subramanian, 2013), whereas credit market disruptions hinder innovation (Hombert and Matray, 2017; Granja and Moreira, 2023).

This body of empirical evidence presents a puzzle in light of the conventional view that debt may not be desirable for financing innovation. Early work argues that debt contracts might not be well-suited for financing innovation due to the uncertainty surrounding research and development (R&D) outcomes, which can lead to credit rationing and premature liquidation (Stiglitz, 1985; Atanassov, Nanda, and Seru, 2007). For practitioners engaged in innovation and its financing, however, it is crucial to distinguish between uncertainty (or ambiguity) and risk. The significant asymmetry of innovation payoffs further poses challenges to aligning fixed-obligation debt with innovation incentives (Manso, 2011). In addition, the intangible nature of R&D constrains collateral value and reduces the debt capacity of innovative firms (Hall and Lerner, 2010). Lastly, why debt can foster innovation remains an open question.

To address this question, we integrate the distinctive features of innovation returns into a framework for financing a growth option, originally developed by Sundaresan and Wang (2007) and Sundaresan, Wang, and Yang (2015). We distinguish risk—characterized by a single known probability distribution—from ambiguity, which is characterized by model uncertainty represented by a set of plausible distributions. In our setting, an entrepreneur makes joint decisions regarding when to initiate an innovation project and how to finance it. Once launched, the project generates cash flows characterized by both jumps and ambiguity—key features of innovation returns that distinguish them from conventional investments, as highlighted by Kerr, Nanda, and Rhodes-Kropf (2014) and Kerr and Nanda (2015). While jumps are intrinsic to the cash flow dynamics under the objective measure, ambiguity reflects the entrepreneur's subjective beliefs (Seo, 2009; Baillon, Huang, Selim, and Wakker, 2018). The entrepreneur can finance the project either entirely with equity (*all-equity financing*) or with an optimal mix of equity and debt (*optimal financing*), the latter leveraging the tax benefits of debt while accounting for bankruptcy costs. Both the entrepreneur and external financiers are ambiguity-averse, making decisions under worst-case scenarios.

Our model highlights the central role of debt in fostering innovation through an *investment acceleration mechanism*. This mechanism emerges from the dynamic interaction between optimal investment timing, capital structure, and cash flow uncertainty. Under complete information, our model jointly and endogenously determines financing and investment decisions in a first-best equilibrium, without relying on exogenous financing constraints.

To ensure analytical tractability, we model innovation cash flows using the double-exponential jump diffusion process introduced by Kou (2002). This process not only captures the leptokurtic features commonly observed in asset and innovation returns, but also allows for a closed-form characterization of the first-passage time distribution. Since both the investment and default options are path-dependent and their values hinge critically on first-passage probabilities, this tractability offers a distinct advantage over alternative jump-size specifications.

We extend the original recursive multiple priors utility (RMPU) framework developed by Chen and Epstein (2002) to incorporate both diffusion and jump ambiguity. In the RMPU framework, agents possess a set of equivalent priors (probability measures) characterized by Itô diffusions and maximize utility under the worst-case prior. The state process under an alternative prior differs only in its drift, thereby altering the first moment only. However, this framework falls short when analyzing innovation as a driving force behind investment and financing choices, because diffusion increments are symmetric and normally distributed.

In this pursuit, we build on the mathematical framework of Quenez and Sulem (2013, 2014) to model agents' multiple priors on innovation-driven cash flows as a set of equivalent Lévy processes. Under both drift and jump ambiguity, the density generator for an alternative prior consists of two components: one for Brownian risk and another for the Poisson random measure. Under jump ambiguity, the state process deviates from the reference measure not just in the mean but across the entire distribution. Solving the optimal stopping problem under such conditions is analytically challenging, particularly in determining the worst-case measure. To address this, we apply the theories of backward stochastic differential equations (BSDEs) with jumps (Quenez and Sulem, 2013, 2014) to derive the worst-case measure and closed-form solutions for optimal investment and financing decisions.

We use detection-error probabilities to calibrate ambiguity and compute relative entropy growth to quantify the contributions of diffusion ambiguity and jump ambiguity, building on Anderson, Hansen, and Sargent (2003), Maenhout (2006), and Aït-Sahalia and Matthys (2019). Under ambiguity, the worst-case measure is considered reasonable when it is statistically difficult to distinguished from the reference measure. Relative entropy growth measures the statistical distance between two priors. In the model, the total relative entropy growth is comprised of two additive components driven by diffusion and jump ambiguity. In the context of innovation, it is important to examine the influence of the two types of ambiguity. For instance, in high-tech sectors where innovations often exhibit radical characteristics, the role of jump ambiguity is naturally magnified, whereas traditional sectors are expected to show the opposite feature.

We show that the investment acceleration benefit of debt for project value arises from the interaction between two effects: the reduction in the investment threshold and the passage time effect—the time it takes for cash flows to reach a target level from below. Under optimal financing, firms invest at a lower cash flow level rather than waiting to reach the higher threshold required under all-equity financing, as the net tax benefit of debt offsets the difference.[1] This lower threshold leads to earlier investment in expectation, thereby increasing the project's expected net present value. Moreover, we demonstrate that the passage time effect amplifies this value gain in a power-law fashion.

Innovation projects, which typically involve greater ambiguity, benefit more from the investment acceleration effect of debt. Although ambiguity raises the investment thresholds, delays investment and lowers project values under both financing scenarios, optimal financing mitigates these adverse effects.[2] On one hand, ambiguity reduces optimal leverage and hence the net tax benefit of debt, narrowing the threshold gap between optimal and all-equity financing.

[1] We also examine a fixed cash flow policy in which firms invest once the cash flow reaches an exogenously determined target. Facing uncertainty in investment returns, firms optimally choose smaller project sizes. We thank the anonymous reviewer for this valuable suggestion.

[2] The effect of ambiguity differs fundamentally from that of volatility, even though both increase the investment threshold and delay investment. Ambiguity reduces the perceived upside potential of the underlying process and the option value due to the monotonicity of the payoff. By contrast, higher volatility increases both upside and downside potential symmetrically, which raises the option value due to the convexity of the payoff.

Nevertheless, this gap shrinks at a diminishing rate. On the other hand, ambiguity strengthens the passage time effect by further delaying investment under all-equity financing. The combination of a slowly declining threshold gap and an increasingly dominant power effect explains why the investment acceleration benefit of debt is amplified under greater ambiguity.

Moreover, our analysis reveals that the fostering effect of debt is particularly pronounced when innovative firms have limited historical data for gauging ambiguity. This result help explain why young startups, which typically lack extensive track records, often favor debt financing. This observation is consistent with empirical evidence from Robb and Robinson (2014), Davis et al. (2020), and Chava et al. (2013). Additionally, we find that the fostering effect of debt is especially significant for radical innovations when jump ambiguity is the primary concern. This finding aligns with Benfratello et al. (2008), who show that high-tech firms benefit greatly from increased credit availability, given their focus on radical innovations capable of generating substantial jumps in returns.

Our theoretical results on the elevated investment threshold and reduced project scale under ambiguity contribute to the broader literature on investment under uncertainty, pioneered by Bloom (2009).[3] Recent studies including Gulen and Ion (2016) and Campello, Cortes, d'Almeida, and Kankanhalli (2022), document that heightened economic policy uncertainty suppresses corporate investment. As recent research (e.g., Aït-Sahalia, Matthys, Osambela, and Sircar, 2025) uses the economic policy uncertainty index as a proxy for ambiguity, these findings align well with the prediction our model that ambiguity delays investment. In addition, our result that greater ambiguity leads to smaller optimal project scales echoes the findings of Campello, Kankanhalli,

[3] Campello and Kankanhalli (2024) provide a comprehensive survey of this literature.

and Kim (2024), who show that firms respond to elevated uncertainty by delaying both investment and disinvestment, particularly when sunk costs are high. Taken together, these empirical patterns reinforce our theoretical insight that ambiguity discourages large-scale commitments in uncertain environments.

Existing real option models focus on diffusion ambiguity only, for example, see Nishimura and Ozaki (2007) and Miao and Wang (2011). These models do not consider jump ambiguity, nor do they investigate financing choices, as we do in our paper. Dicks and Fulghieri (2021) develop a theory of innovation waves and investor sentiment tied to ambiguity. Coiculescu, Izhakian, and Ravid (2024) treat innovation as real options and find that ambiguity negatively impacts R&D, while risk has a positive effect. A recent and closely related study by Geelen, Hajda, and Morellec (2022) develop a Schumpeterian growth model with endogenous R&D and financing choices, demonstrating that debt fosters innovation and growth at the aggregate level, similar to our findings. They attribute this effect to the tax benefit of debt. Different from this study, our model emphasizes the interaction between debt financing and jump ambiguity, a defining feature of innovation returns.

II. The Model

Our baseline model combines the irreversible investment framework of McDonald and Siegel (1986) with the EBIT-based capital structure model of Goldstein, Ju, and Leland (2001). Sundaresan and Wang (2007) and Sundaresan et al. (2015) formally analyze the joint implications of these models and extend them further. We adopt their assumptions but introduce jump risk into the baseline model. Our main contribution is to examine the effects of ambiguity about both the drift and the jump intensity and size distribution.

A. The Baseline Model without Ambiguity

The project. An entrepreneur has access to an innovation project that, once initiated, generates an earnings before interest and taxes (EBIT) flow $X(t) := X(t, \omega)$ defined on a probability space $(\Omega, \mathcal{F}, Q^0)$ endowed with a standard complete filtration $\mathbf{F} = \{\mathcal{F}_t | t \geq 0\}$ satisfying the "usual conditions". We assume that the EBIT process is exogenous and independent of the entrepreneur's investment decision. We specify Q^0 to be the risk-neutral probability measure, in line with the traditional EBIT-based capital structure model of Goldstein et al. (2001).[4] Under Q^0, $X(t)$ follows a geometric Lévy process:

$$\frac{dX(t)}{X(t^-)} = \mu dt + \sigma dW(t) + \int_{\mathbb{R}} \iota(t, u) \tilde{N}(dt, du), \tag{1}$$

where $W(t)$ is a standard Brownian motion, $\tilde{N}(dt, du)$ is a compensated Poisson random measure given by

$$\tilde{N}(dt, du) = N(dt, du) - \nu(du)dt,$$

with $\nu(du) := \mathbb{E}[N(1, du)]$ being the Lévy measure, and $\iota(t, u)$ is a square-integrable predictable process with respect to $\nu(du)$. It is worth noting that the specification of a compensated Poisson random measure under the risk-neutral reference measure follows the same rational expectation

[4] Goldstein et al. (2001) specify the EBIT process under the physical measure and define an equivalent risk-neutral measure by adjusting the market price of diffusion risk. Under this risk-neutral measure, the value of an unlevered firm's claim, when discounted at the risk-free rate, is a martingale.

equilibrium pricing rule as in Goldstein et al. (2001).[5] Additionally, we assume that $X(0) = x > 0$, μ, σ, and r are constants with $\mu < r$ and r being the risk-free rate.

We specify the jump component as a compound Poisson process with intensity $\lambda < +\infty$ and assume a double exponential distribution for the jump size, i.e., $X(t)$ follows a double exponential jump-diffusion process, first introduced by Kou (2002). Specifically, we can write the jump component explicitly as

$$\iota(t, u) = e^u - 1, \quad \text{and} \quad \nu(du) = \lambda f(du),$$

where the jump size density f is given by

$$f_u = p\eta_1 e^{-\eta_1 u}\mathbf{1}_{u \geq 0} + q\eta_2 e^{\eta_2 u}\mathbf{1}_{u<0}, \quad \eta_1 > 1,\ \eta_2 > 0,\ p,\ q \geq 0,\ p + q = 1 \tag{2}$$

in which $\mathbf{1}_{\cdot}$ is an indicator function. In the specification, p and q are the conditional probabilities of upward and downward jumps respectively, and $1/\eta_1$ and $-1/\eta_2$ are the conditional mean log jump sizes of positive and negative jumps respectively. This model specification has two merits. First, as shown by Kou and Wang (2004), it allows for analytical solutions to valuation problems

[5] Analogously, one can specify the EBIT dynamics under the physical measure, as in Goldstein et al. (2001). For the jump component, the uncompensated form in Kou (2002) can be adopted. Kou (2002) construct a stochastic discount factor process in a representative-agent rational expectations equilibrium and obtain an implied risk-neutral measure when endowments follow a double exponential jump-diffusion process. Using the stochastic discount factor derived therein, one can recover the risk-neutral EBIT dynamics, which match those in equation (1). Under this measure, the value of an unlevered firm's claim, when discounted at the risk-free rate, remains a martingale.

with American-style perpetual options.[6] This property is crucial to our analyses of joint investment and financing decisions in the real options model. Second, the specification has the appealing feature of modeling positive and negative jumps separately, which generates a rich set of priors, as will be shown later.

Financing. We assume that the project requires external funding, as available cash is insufficient. Taxation on project cash flows at rate $\phi \in (0,1)$ motivates firms to issue debt for tax shielding. The entrepreneur chooses between *all equity financing* and *optimal financing*, aiming to determine the optimal equity-debt mix.[7] Drawing on the established literature (e.g., Leland, 1998; Goldstein et al., 2001), we analyze debt contracts with perpetual coupon payments C in a time-homogeneous context. Debt issuance reduces tax by ϕC but also exposes the firm to potential bankruptcy costs.

We begin with the all-equity financing case. The equity value at the $\mathcal{F}_0$-measurable random investment time $\tau_I \geq 0$ is

$$V_e(\tau_I) = \mathbb{E}_{\tau_I}\Big[\int_{\tau_I}^{\infty} e^{-r(t-\tau_I)}(1-\phi)X(t)dt - I\Big], \tag{3}$$

where the constant $I > 0$ denotes the investment cost. At time 0, the entrepreneur chooses the

[6]For further discussion on the analytical tractability of first-passage time distributions under alternative Lévy measure specifications, see Kou and Wang (2003) and Bakshi and Panayotov (2010).

[7]It is worth noting that optimal financing might coincide with all-equity financing. We leave all possibilities open for now and allow the model to determine whether such a corner solution arises.

optimal stopping time τ_I^e to initiate the project, hence her value function at time 0 is

$$V_e(0) = \sup_{\tau_I} \mathbb{E}_0[e^{-r\tau_I} V_e(\tau_I)] = \mathbb{E}_0[e^{-r\tau_I^e} V_e(\tau_I^e)].$$

[Insert Figure 1 approximately here]

In the case of optimal financing, the entrepreneur can issue debt at the time of investment, τ_I, to take advantage of the tax benefits associated with debt. Default occurs later, at the stopping time τ_D under the assumption of optimal default, a standard assumption in the literature (Strebulaev and Whited, 2012; Sundaresan and Wang, 2007; Sundaresan et al., 2015). As per the traditional trade-off theory of capital structure (e.g., Leland, 1994; Goldstein et al., 2001), default triggers liquidation under the absolute priority rule, with a fraction $\alpha \in (0, 1)$ of the asset value lost in the liquidation process. This assumption is standard in models that balance the benefits of debt (via tax shields) with the costs of potential default and liquidation. Figure 1 illustrates the timelines for both financing choices.

Given the above, the firm value at the investment time τ_I is

$$V_*(\tau_I) = E(\tau_I) + D(\tau_I) - I,$$

where $D(\tau_I)$ is the entrepreneur's valuation of debt, and $E(\tau_I)$ is the value of (levered) equity, given by

$$\begin{aligned} E(\tau_I) &= \sup_{\tau_D \geq \tau_I} \mathbb{E}_{\tau_I}\Big[\int_{\tau_I}^{\tau_D} e^{-r(t-\tau_I)}(1-\phi)\big(X(t)-C\big)dt\Big] \\ &= \mathbb{E}_{\tau_I}\Big[\int_{\tau_I}^{\tau_D^*} e^{-r(t-\tau_I)}(1-\phi)\big(X(t)-C\big)dt\Big]. \end{aligned} \tag{4}$$

The above indicates that the equity value is the discounted net profit received by equity holders until default. At default, equity holders recover nothing. As is standard in the trade-off theory of capital structure, we assume that the entrepreneur maximizes equity value by choosing the default policy τ_D. However, when choosing the optimal debt policy C^*, she maximizes firm value $V_*(\tau_I)$, incorporating her expectation of debt value $D(\tau_I)$.[8] In a symmetric rational expectation equilibrium, the entrepreneur's valuation of debt, $D(\tau_I)$, equals debt investors' valuation and is given by

$$D(\tau_I) = \mathbb{E}_{\tau_I}\Big[\int_{\tau_I}^{\tau_D^*} e^{-r(t-\tau_I)} C dt + (1-\alpha)\int_{\tau_D^*}^{\infty} e^{-r(t-\tau_I)}(1-\phi)X(t)dt\Big].$$

The first term inside the conditional expectation operator is the present value of the coupon collected while the firm remains solvent. The second term is the present value of the asset value recovered upon liquidation in bankruptcy.

Since τ_D^* and C^* depend on τ_I^*, these three decision variables are jointly determined at time 0 by maximizing the discounted firm value at investment

$$V_*(0) = \sup_{\tau_I, \tau_D > \tau_I, C} \mathbb{E}_0[e^{-r\tau_I} V_*(\tau_I)] = \sup_{\tau_I, \tau_D > \tau_I, C} \mathbb{E}_0[e^{-r\tau_I}\mathbb{E}_{\tau_I}[V_*(\tau_I)]],$$

where the second equality follows from the property of conditional expectations, and the optimization is subject to the default policy determined in (4).

[8] Strebulaev and Whited (2012) note that equity holders do not internalize debt value when making a default decision, which generates the *ex ante* vs. *ex post* conflict of interest. They further note that the *ex ante* commitment for equity holders to maximize firm value at every point in time is not contractible.

B. The Set of Priors

Agents face ambiguity about the EBIT reference model in equation (1) and consider "close" alternative models. With ambiguity aversion, they opt for worst-case EBIT dynamics. To capture jump ambiguity, we extend the utility framework of Chen and Epstein (2002) allow for uncertainty in both jump intensity and size distribution. Our extension builds on the general results of Quenez and Sulem (2013, 2014), who provide the comparison theorem for BSDEs under Lévy processes and general results for related optimal stopping problems under ambiguity.

Let Θ denote the set of density generators. Each density generator $\theta \in \Theta$ is binary, i.e., $\theta = (\theta_W, \theta_N)$, where θ_W is for the Brownian motion and θ_N for the jump component. For each $\theta \in \Theta$, let $Z^\theta(t)$ be the solution to the (forward) SDE:

$$dZ^\theta(t) = Z^\theta(t^-)\Big(- \theta_W(t)dW(t) - \int_{\mathbb{R}} \theta_N(t,u)d\tilde{N}(dt,du)\Big),$$

for $t \in [0,T]$ with $Z^\theta(0) = 1$. For $\theta_W(t)$, we adopt the κ-ignorance specification of Chen and Epstein (2002), i.e., $\theta_W(t) \in [-\kappa, \kappa], 0 < \kappa < \infty$. Based on the technical requirements in Quenez and Sulem (2013, 2014), we specify $\theta_N(t,u)$ as:

$$\theta_N(t,u) = 1 - e^{\theta_{N,1}(t)u}\mathbf{1}_{u\geq 0} - e^{\theta_{N,2}(t)u}\mathbf{1}_{u<0},$$

$$\theta_{N,1}(t) \in [-M_1, 0], \quad \theta_{N,2}(t) \in [0, M_2], \quad \text{and} \quad M_1, M_2 > 0.$$

This specification for $\theta_N(t,u)$ follows from the jump-size distribution in equation (2), and we discuss its implications after constructing the set of priors.

To construct the set of priors, we define a probability measure Q^θ on $\mathcal{F}_T$, equivalent to Q^0

for $\theta \in \Theta$ as

$$\mathbb{E}^{\theta}[\mathbf{1}_A] = \mathbb{E}[\mathbf{1}_A Z_T^{\theta}], \quad A \in \mathcal{F}_T.$$

Hence, under Q^{θ},

$$dW^{\theta}(t) = dW(t) + \theta_W(t)dt,$$

which represents the Brownian risk, and

$$\tilde{N}^{\theta}(dt, du) = \tilde{N}(dt, du) + \theta_N(t, u)\nu(du)dt = N(dt, du) - (1 - \theta_N(t, u))\nu(du)dt, \tag{5}$$

which is a a compensated Poisson random measure by Girsanov's theorem; see Øksendal and Sulem (2019, Chapter 1.4). Taken together, under Q^{θ}, $X(t)$ is given by

$$\frac{dX(t)}{X(t^-)} = \Big(\mu - \theta_W(t)\sigma - \int_{\mathbb{R}} (e^u - 1)\theta_N(t, u)\nu(du)\Big)dt + \sigma dW^{\theta}(t) + \int_{\mathbb{R}} (e^u - 1)\tilde{N}^{\theta}(dt, du).$$

Our method of distorting the reference measure can attenuate the impact of positive or negative jumps, depending on the monotonicity of the entrepreneur's value function in the state variable. If the value function increases with the state variable, as in the baseline model without ambiguity, the worst-case measure leads to the smallest unconditional probability and mean size for positive jumps, and the largest values for negative jumps—all constrained within the set of priors. To see this, (5) shows that $N^{\theta}(dt, du)$ has a Lévy measure $\nu^{\theta}(du) = (1 - \theta_N(t, u))\nu(du)$ under Q^{θ}. We can express $\nu^{\theta}(du)$ as

$$\nu^{\theta}(du) = \lambda_t^{\theta} f_t^{\theta}(du),$$

where λ_t^θ is the distorted jump intensity given by

$$\lambda_t^\theta = \lambda \int_{\mathbb{R}} \big(1-\theta_N(t,u)\big) f_u du = \lambda\Big(\frac{p\eta_1}{\eta_1-\theta_{N,1}(t)} + \frac{q\eta_2}{\eta_2+\theta_{N,2}(t)}\Big),$$

and the distorted jump size density is given by

$$\begin{aligned} f_{u,t}^\theta &= \frac{\big(1-\theta_N(t,u)\big) f_u}{\int_{\mathbb{R}} \big(1-\theta_N(t,u)\big) f_u du} \\ &= p_t^\theta\big(\eta_1-\theta_{N,1}(t)\big)e^{-(\eta_1-\theta_{N,1}(t))u}\mathbf{1}_{u\geq 0} + q_t^\theta\big(\eta_2+\theta_{N,2}(t)\big)e^{(\eta_2+\theta_{N,2}(t))u}\mathbf{1}_{u<0}, \end{aligned}$$

with p_t^θ and q_t^θ being the distorted probabilities of upward and downward jumps given by

$$p_t^\theta = \frac{p\eta_1\big(\eta_2+\theta_{N,2}(t)\big)}{p\eta_1\big(\eta_2+\theta_{N,2}(t)\big)+q\eta_2\big(\eta_1-\theta_{N,1}(t)\big)} \text{ and } q_t^\theta = \frac{q\eta_2\big(\eta_1-\theta_{N,1}(t)\big)}{p\eta_1\big(\eta_2+\theta_{N,2}(t)\big)+q\eta_2\big(\eta_1-\theta_{N,1}(t)\big)}.$$

[Insert Table 1 approximately here]

Table 1 summarizes the Lévy measures for $N^\theta(dt,du)$ under the set of priors specified $\theta_{N,1}(t)\in[-M_1,0]$ and $\theta_{N,2}(t)\in[0,M_2]$. In the extreme case where only positive jumps occur ($p=1$), the jump intensity ranges from $[\lambda\eta_1/(\eta_1+M_1),\lambda]$, with the conditional probability of positive jumps fixed at 1. If the worst-case scenario corresponds to $\theta_{N,1}(t)=-M_1$ (or 0) for all $t\geq 0$, the jump intensity is minimized (or maximized). A similar observation holds for the other extreme case ($q=1$).

C. Innovation with All-Equity Financing under Ambiguity

With the set of priors, the entrepreneur's value function under all-equity financing at the investment time τ_I and under a candidate measure Q^{θ} is

$$V_e(\tau_I;\theta) = \mathbb{E}^{\theta}_{\tau_I}\Big[\int_{\tau_I}^{\infty} e^{-r(t-\tau_I)}(1-\phi)X(t)dt - I\Big]. \tag{6}$$

At time 0, she chooses the optimal investment time τ_I^e to start the project. The ambiguity-averse entrepreneur chooses the worst-case measure Q^{θ^*} from the set of priors. As long as $Q^{\theta^e}\{\tau_I^* < +\infty\} = 1$, her value function at time 0 is

$$V_e(0;\tau_I^e,\theta^*) = \sup_{\tau_I \geq 0}\ \inf_{Q^{\theta}}\ \mathbb{E}^{\theta}_0\big[e^{-r\tau_I}V_e(\tau_I;\theta)\big]. \tag{7}$$

It is necessary to discuss a few technical details of the optimal stopping problem under ambiguity, given by equations (6) and (7). First, Quenez and Sulem (2014) prove that the nonlinear expectation in (7) admits the minmax relation. Thus, we can first solve the minimum expectation problem to find the worst-case measure Q^{θ^*} and then solve the optimal stopping problem for τ_I^* under Q^{θ^*}. Because the minimum expectation problem involves the application of Girsanov's theorem and the comparison theorem for BSDEs under Lévy processes and because Q^{θ^*} exists for any horizon in our model, we can solve the optimal stopping problem under Q^{θ^*} for the infinite horizon. Second, Quenez and Sulem (2013) show that the minimum expectation is dynamically consistent:

$$\inf_{Q^{\theta}}\ \mathbb{E}^{\theta}_0\big[e^{-r\tau_I}V_e(\tau_I;\theta)\big] = \inf_{Q^{\theta'}}\ \mathbb{E}^{\theta'}_0\Big[\inf_{Q^{\theta''}}\ \mathbb{E}^{\theta''}_{\tau_I}\big[e^{-r\tau_I}V_e(\tau_I;\Big\{\theta',\theta''\Big\}))\big]\Big].$$

where θ' and θ'' deliver density generators for the decision intervals $[0, \tau_I]$ and $[\tau_I, T]$ with $T \leq +\infty$, respectively. Dynamic consistency provides analytical convenience in that the worst-case density generators for $[0, \tau_I]$ and $[\tau_I, T]$ coincide with that for $[0, T]$. Hence, it suffices to begin with the following to find the worst-case measure:

$$V_e(0; \tau_I, \theta^*) = \inf_{Q^\theta} \mathbb{E}_0^\theta\big[e^{-r\tau_I} V_e(\tau_I; \theta)\big], \quad \text{for any } \tau_I. \tag{8}$$

Proposition 1. *The density generator that gives the minimum expectation in (8) is* $\theta^* = (\theta_W^*, \theta_N^*) = (\kappa, 1 - e^{-M_1 u}\mathbf{1}_{u\geq 0} - \mathbf{1}_{u<0})$ *for all* $t \in [0, T]$.

In the Internet Appendix, we provide the proof, which relies on dynamic consistency and the comparison theorem for BSDEs under Lévy process.

An immediate implication is that under Q^{θ^*}, $X(t)$ follows the process

$$\frac{dX(t)}{X(t^-)} = \mu^{\theta^*} dt + \sigma dW^{\theta^*}(t) + \int_{\mathbb{R}} (e^u - 1)\tilde{N}^{\theta^*}(dt, du)$$

where

$$\begin{aligned}\mu^{\theta^*} &= \mu - \kappa\sigma - \int_{\mathbb{R}} (e^u - 1)(1 - e^{-M_1 u}\mathbf{1}_{u\geq 0} - \mathbf{1}_{u<0})\nu(du) \\ &= \mu + \underbrace{(-\kappa\sigma)}_{\text{drift ambiguity discount}<0} + \underbrace{\Big(-\frac{\lambda p}{(\eta_1 - 1)} + \frac{\lambda p \eta_1}{(\eta_1 + M_1 - 1)(\eta_1 + M_1)}\Big)}_{\text{jump ambiguity discount}<0}.\end{aligned}$$

The distorted process above indicates that drift ambiguity and jump ambiguity reduce the perceived mean return by $\kappa\sigma$ and $\lambda p/(\eta_1 - 1) - \lambda p \eta_1/(\eta_1 + M_1 - 1)(\eta_1 + M_1)$, respectively.

Before turning to the detailed discussion of $X(t)$ under Q^{θ^*}, it's important to note that jump ambiguity distortions remain independent of drift ambiguity. This property arises from the independence of the Brownian and jump terms in line with the Itô-Lévy Decomposition. In essence, the relative importance of the two drift distortions depends on the entrepreneur's relative concern with diffusion ambiguity versus jump ambiguity.

Importantly, beyond reducing drift, jump ambiguity also distorts jump intensity and the size distribution. For instance, the conditional mean for positive jumps (in log units) reaches its minimum at $1/\eta_1^*$, where $\eta_1^* = \eta_1 + M_1$, while the conditional mean of negative jumps (in log units) attains its maximum at $-1/\eta_2^*$, with $\eta_2^* = \eta_2$. The conditional probability of positive jumps reaches its minimum at $p^* = p\eta_1/(p\eta_1 + q(\eta_1 + M_1))$, and that of negative jumps achieves its maximum at $q^* = q(\eta_1 + M_1)/(p\eta_1 + q(\eta_1 + M_1))$. The jump intensity in the worst-case scenario does not attain either its maximum or minimum value, but instead equals $\lambda^* = \lambda(p\eta_1/(\eta_1 + M_1) + q)$.

These worst-case parameters are summarized in Table 1. The interpretation of these results follows from our earlier discussion of Table 1, where we considered the monotonicity of the value function with respect to the state variable. Specifically, the parameter λ^* takes an interior value, as the distortion minimizes the unconditional probability of positive jumps while maximizing that of negative jumps.

The expression for $X(t)$ under Q^{θ^*} reveals that jump ambiguity distorts the entire distribution of $X(t)$, whereas drift ambiguity distorts the mean only. This is evident by examining the moment generating function of $Y(t) = \ln(X(t)/X(0))$ as

$$\mathbb{E}^{\theta^*}\left[e^{vY(t)}\right] = e^{t\phi^{\theta^*}(v)},$$

where

$$\phi^{\theta^*}(v) = \frac{1}{2}\sigma^2 v^2 + \Big(\mu^{\theta^*} - \frac{1}{2}\sigma^2 - \lambda^*\Big(\frac{p\eta_1^*}{\eta_1^* - 1} + \frac{q^*\eta_2^*}{\eta_2^* + 1} - 1\Big)\Big)v + \lambda^*\Big(\frac{p^*\eta_1^*}{\eta_1^* - v} + \frac{q^*\eta_2^*}{\eta_2^* + v} - 1\Big).$$

It follows immediately that the variance of $Y(t)$ is

$$\mathbb{E}^{\theta^*}\left[\left(Y(t) - \mathbb{E}^{\theta^*}[Y(t)]\right)^2\right] = \sigma^2 t + 2\lambda^*\Big(\frac{p^*}{(\eta_1^*)^2} + \frac{q^*}{(\eta_2^*)^2}\Big)t.$$

In contrast to drift ambiguity, jump ambiguity has a significant effect on the variance of $Y(t)$ and higher moments, which we demonstrate numerically in Section III.B.

After finding the worst-case measure, we solve for the optimal investment policy and the value function in equation (7). Here, we can gain analytical tractability of the double exponential jump-diffusion process for optimal stopping problems. The following proposition characterizes the optimal investment policy and the associated value function under all-equity financing.

Proposition 2. *Let $\eta_1^* = \eta_1 + M_1$ and $\eta_2^* = \eta_2$. For $X(0) = x$, the value function $V_e(0; \tau_I^e, \theta^*)$ in (7) has the solution*

$$V_e(0; \tau_I^e, \theta^*) = A_0 X_I^e\Big[c_{1,1}\Big(\frac{X_I^e}{x}\Big)^{-\beta_1} + c_{2,1}\Big(\frac{X_I^e}{x}\Big)^{-\beta_2}\Big] - I\Big[c_{1,0}\Big(\frac{X_I^e}{x}\Big)^{-\beta_1} + c_{2,0}\Big(\frac{X_I^e}{x}\Big)^{-\beta_2}\Big],$$

where the optimal stopping time satisfies $\tau_I^e = \inf_t \{X(t) \geq X_I^e\}$, and the optimal investment boundary X_I^e is given by

$$A_0 X_I^e = \underbrace{\frac{\beta_1\beta_2}{(\beta_1 - 1)(\beta_2 - 1)}\frac{\eta_1^* - 1}{\eta_1^*}}_{\text{option multiplier}>1} I, \quad A_0 = \underbrace{\frac{1 - \phi}{r - \mu^{\theta^*}}}_{\text{value multiplier}}. \tag{9}$$

Here, β_1*,* β_2*,* β_3*,* β_4 *are constants satisfying*

$-\infty < -\beta_4 < -\eta_2^* < -\beta_3 < 0 < \beta_1 < \eta_1^* < \beta_2 < \infty$ *and are the four roots of the equation* $G(\beta) = r$*, with*

$$(10)\ \ G(\beta) = \frac{1}{2}\sigma^2\beta^2 + \Big[\mu^{\theta^*} - \frac{1}{2}\sigma^2 - \lambda^*\Big(\frac{p^*\eta_1^*}{\eta_1^* - 1} + \frac{q^*\eta_2^*}{\eta_2^* + 1} - 1\Big)\Big]\beta + \lambda^*\Big(\frac{p^*\eta_1^*}{\eta_1^* - \beta} + \frac{q^*\eta_2^*}{\eta_2^* + \beta} - 1\Big).$$

The constants $c_{1,0}, c_{2,0}, c_{1,1}$ *and* $c_{2,1}$ *are given by*

$$(11)\qquad c_{1,0} = \frac{\eta_1^* - \beta_1}{\beta_2 - \beta_1}\frac{\beta_2}{\eta_1^*},\ c_{2,0} = \frac{\beta_2 - \eta_1^*}{\beta_2 - \beta_1}\frac{\beta_1}{\eta_1^*},\ c_{1,1} = \frac{\eta_1^* - \beta_1}{\beta_2 - \beta_1}\frac{\beta_2 - 1}{\eta_1^* - 1},\ c_{2,1} = \frac{\beta_2 - \eta_1^*}{\beta_2 - \beta_1}\frac{\beta_1 - 1}{\eta_1^* - 1}.$$

The investment payoff at τ_I^e is $A_0X(\tau_I^e) - I$. In equation (9), the term $A_0X_I^e$ represents the equity value at τ_I^e (in the absence of overshooting) and admits a Gordon growth-like interpretation.[9] The value multiplier (VM) reflects the project's value when $X(t) = 1$, and it decreases as ambiguity increases. On the right-hand side, the equity value is expressed in terms of the cost I and the option multiplier (OM), which accounts for the effect of uncertainty on the investment decision. Here, uncertainty includes both diffusion and jump risks, characterized by the primitive parameters of the double-exponential jump-diffusion process, as well as drift and jump ambiguity.

Remark 1. *Jump ambiguity reduces the project value by lowering the investment payoff at* τ_I^e*, raising the cash flow threshold* X_I^e*, and extending the expected threshold hitting time* τ_I^e*.*

[9]Because jumps make the sample path of $X(t)$ discontinuous, it is possible for $X(t)$ to overshoot, i.e., $X(\tau_I^e) \geq X_I^e$. For more details, see the proof of Proposition 2 and Kou and Wang (2003).

An immediate observation from the inequality:

$$\frac{\beta_2}{\beta_2 - 1} I < A_0 X_I^e < \frac{\beta_1}{\beta_1 - 1} I,$$

is that as jump ambiguity $M_1 \to \infty$, β_2 approaches infinity, causing the lower bound of $A_0 X_I^e$ to decrease monotonically toward I. Although the limit of β_1 is harder to deduce, it is evident that the upper bound also decreases monotonically with M_1. We conjecture that both bounds eventually converge, reducing the OM to 1 and driving the equity value at τ_I^e to the net present value (NPV) threshold, I.

We deduce that X_I^e increases with M_1. While the effect of M_1 on X_I^e is not immediately clear due to the simultaneous decrease in both the OM and VM, we can infer the relationship by comparing their rates of change. Computing $\partial X_I^e / \partial M_1$ is technically challenging, but the decline in μ^{θ^*} with M_1 follows a rate bounded by $\tilde{C}/M_1^2$, leading the VM to decrease at the same rate, where $\tilde{C} > 0$ denotes an arbitrary constant. Moreover, the upper and lower bounds for $A_0 X_I^e$ decrease at a rate bounded by $\tilde{C}/M_1$. Thus, we numerically verify that X_I^e increases with M_1 at a rate bounded by $\tilde{C}/M_1$.

Lastly, we anticipate that the expected investment time, τ_I^e, also increases with M_1 for two reasons. First, investment occurs when $X(t)$ reaches X_I^e from below, and under Q^{θ^*}, $X(t)$ has the least upward potential, leading to a longer average time to reach any given level compared to Q^0. Second, since X_I^e is higher under Q^{θ^*} than under Q^0, the expected hitting time is further extended, reinforcing the delay in investment.

Remark 2. *Under a fixed cash flow threshold policy, ambiguity induces the firm to undertake smaller innovation projects.*

A fixed cash flow threshold policy is to invest when $X(t) \geq \bar{X}$ for the first time, where $\bar{X}$ is a fixed level.[10] In this case, the optimality condition (9) implies that the firm can maximize its equity value by choosing I optimally, satisfying $I_e^* = \bar{X} \times VM/OM$. As discussed above, jump ambiguity lowers the VM at a rate bounded by $\tilde{C}/M_1^2$ and the OM at a rate bounded by $\tilde{C}/M_1$. Hence, I_e^* decreases with M_1 at a rate bounded by $\tilde{C}/M_1$.

Remark 2 parallels the insight of Campello and Kankanhalli (2024), who show that greater uncertainty reduces investment size in a two-period framework with mean-preserving spread.[11] Our approach differs from that of Campello and Kankanhalli (2024), as in that model the ambiguity-neutral decision maker evaluates uncertainty through second-order stochastic dominance, whereas our continuous-time framework explicitly incorporates both diffusion and jump risks together with ambiguity.

D. Innovation with Optimal Financing

1. Optimal Default under Ambiguity

Because of the property $Q^{\theta^*}\{\tau_I^* < +\infty\} = 1$ in the all-equity financing case and dynamic consistency, we can first solve for the entrepreneur's optimal default decision and then the optimal coupon to determine the optimal capital structure. Under the worst-case measure Q^{θ^*} (Proposition 3), $Q^{\theta^*}\{\tau_D^* < +\infty\} = 1$ holds for our parameter requirements, and thus the levered equity value

[10] We thank the anonymous reviewer for this perspective.

[11] Their model builds on the framework of Campello et al. (2022) and Campello et al. (2024), where project size is the firm's optimal choice variable. They emphasize that the main advantage of using mean-preserving spread lies in its generality, as it does not require specifying the exact distribution of outcomes.

at τ_I follows

$$E(\tau_I; \tau_D^*, \theta^*, C) = \sup_{\tau_D \geq \tau_I} \inf_{Q^\theta} \mathbb{E}_{\tau_I}\Big[\int_{\tau_I}^{\tau_D} e^{-r(t-\tau_I)}(1-\phi)\big(X(t)-C\big)dt\Big]. \tag{12}$$

Without ambiguity, the problem reduces to the baseline model in equation (4).

Using the same approach as in Section II.C, we begin by finding the density generator that delivers the worst-case measure:

$$E(\tau_I; \tau_D, \theta^*, C) = \inf_{Q^\theta} \mathbb{E}_{\tau_I}\Big[\int_{\tau_I}^{\tau_D} e^{-r(t-\tau_I)}(1-\phi)\big(X(t)-C\big)dt\Big]. \tag{13}$$

Proposition 3. *The density generator that gives the minimum expectation in (13) is* $\theta^* = (\theta_W^*, \theta_N^*) = (\kappa, 1 - e^{-M_1 u}\boldsymbol{1}_{u\geq 0} - \boldsymbol{1}_{u<0})$ *for all* $t \in [\tau_I, T]$.

Proposition 3 establishes that the worst-case measure is the same under both optimal financing and all-equity financing. This result follows because, in both cases, equity holders receive net profit, and their main concern under ambiguity is that the perceived EBIT profile is the least favorable.

Given that we have determined the worst-case prior for equity holders, we can then solve for the optimal default timing τ_D^* and the value function. The results are summarized in the proposition below.

Proposition 4. *Let* β_3*,* β_4*,* μ^{θ^*}*,* η_1^**,* η_2^**, and* A_0 *be the same as in Proposition 2. The levered equity* *value $E(\tau_I; \tau_D^*, \theta^*, C)$ in (12) has the following expression*

$$E(\tau_I; \tau_D^*, \theta^*, C) = A_0 X(\tau_I) - \frac{(1-\phi)C}{r}\Big[1 - d_{1,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3} - d_{2,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big]$$
$$- A_0 X_D^*\Big[d_{1,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3} + d_{2,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big],$$

where

$$d_{1,0} = \frac{\eta_2^* - \beta_3}{\beta_4 - \beta_3}\frac{\beta_4}{\eta_2^*},\ d_{2,0} = \frac{\beta_4 - \eta_2^*}{\beta_4 - \beta_3}\frac{\beta_3}{\eta_2^*},\ d_{1,1} = \frac{\eta_2^* - \beta_3}{\beta_4 - \beta_3}\frac{\beta_4 + 1}{\eta_2^* + 1},\ d_{2,1} = \frac{\beta_4 - \eta_2^*}{\beta_4 - \beta_3}\frac{\beta_3 + 1}{\eta_2^* + 1}, \tag{14}$$

and the optimal default policy is $\tau_D^* = \inf_t\{X(t) \leq X_D^*\}$ with the optimal default boundary X_D^* given by

$$A_0 X_D^* = \underbrace{\frac{\beta_3\beta_4(\eta_2^* + 1)}{(\beta_3 + 1)(\beta_4 + 1)\eta_2^*}}_{\text{option multiplier}<1}\frac{(1-\phi)C}{r}. \tag{15}$$

At default, under liquidation bankruptcy, equity holders lose the firm, valued at $A_0 X_D^*$—the left-hand side of equation (15). At the same time, they cease paying the tax deductible perpetual coupons, valued at $(1-\phi)C/r$. Thus, the NPV rule dictates that default occurs when $A_0 X(t) = (1-\phi)C/r$ for the first time. However, defaulting at the NPV threshold is suboptimal because equity holders have the option to inject additional equity to delay default, anticipating a future rebound in $X(t)$. Therefore, the optimal default threshold is set below the NPV threshold, discounted by the OM, which reflects the factors discussed earlier in relation to X_I^* in Proposition 2.[12]

[12] Another plausible default scenario is liquidity default, where default is triggered when $X(t)$ reaches C for the first time. As noted by Strebulaev and Whited (2012), liquidity default mainly affects optimal leverage. In our

Remark 3. *Jump ambiguity has a secondary effect on optimal default boundary* X_D^*.

For X_D^*, we can establish its lower and upper bounds as follows:

$$\frac{\beta_3+1}{\beta_3}\frac{(1-\phi)C}{r} < A_0 X_D^* < \frac{\beta_4+1}{\beta_4}\frac{(1-\phi)C}{r}.$$

Since $\theta_{N,2}=0$ under Q^{θ^*}, jump ambiguity governed by M_1 does not significantly affect β_3 and β_4, implying that the bounds for $A_0X_D^*$ do not converge in the same way as the bounds for $A_0X_I^*$. As a result, the influence of jump ambiguity on X_D^* stems primarily from the interaction between the VM and C^*, the endogenously determined optimal coupon. This suggests that both the OM and the default boundary are less sensitive to jump ambiguity than their counterparts for investment.

2. Optimal Financing Choice

To find the optimal financing choice, we seek the debt value or optimal coupon that maximizes firm value upon debt issuance. We consider a symmetric equilibrium in which debt investors share the same preferences and set of priors as equity investors. Furthermore, debt and equity holders are mutually aware of each other's decision rules. Thus, for any investment policy τ_I and coupon policy C chosen at the stopping time τ_I, debt holders' valuation is given by

$$(16)\quad D(\tau_I;\tau_D^*,\theta^*,C)=\inf_{Q^\theta}\ \mathbb{E}_{\tau_I}^{\theta}\Big[\int_{\tau_I}^{\tau_D^*}e^{-r(t-\tau_I)}Cdt+(1-\alpha)\int_{\tau_D^*}^{\infty}e^{-r(t-\tau_I)}(1-\phi)X(t)dt\Big].$$

analysis, we focus on optimal default, which has been the dominant assumption in the existing literature (e.g., Goldstein et al., 2001; Strebulaev, 2007; Sundaresan, 2013, and others).

Proposition 5. *The density generator that gives the minimum expectation in (16) is* $\theta^* = (\theta_W^*, \theta_N^*) = (\kappa, 1 - e^{-M_1 u}\boldsymbol{1}_{u \geq 0} - \boldsymbol{1}_{u<0})$ *for all* $t \geq \tau_I$*. Furthermore, the value of debt has the expression*

$$D(\tau_I; \tau_D^*, \theta^*, C) = \frac{C}{r}\Big[1 - d_{1,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3} - d_{2,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big] + A_1 X_D^*\Big[d_{1,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3} + d_{2,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big],$$

where $d_{i,j}$ *is given by (14),* X_D^* *is given by (15), and* $A_1 = (1-\alpha)A_0$*.*

The result above suggests that debt investors use the same pricing measure as the entrepreneur and external equity investors do. This alignment is intuitive: in liquidation, debt investors would effectively become equity holders of the unleveraged firm, capturing the stochastic cash flows produced by the project. In addition, the debt value prior to default, which is represented by the first term in the integrand in equation (16), increases with the state variable. Consequently, their worst-case scenario coincides with that of the equity holders.

Now we can find the optimal capital structure by choosing the optimal coupon C^* to maximize the firm value at the investment time τ_I,

$$V_*(\tau_I; \tau_D^*, \theta^*, C^*) = \max_C \big\{\big(E(\tau_I; \tau_D^*, \theta^*, C) + D(\tau_I; \tau_D^*, \theta^*, C) - I\big)\big\}. \tag{17}$$

Unlike in the pure diffusion case (e.g., Sundaresan and Wang, 2007; Sundaresan et al., 2015), a closed-form solution for C^* is not available for Problem (17). Nevertheless, the following Proposition shows that C^* remains a linear function of $X(\tau_I)$ as in the pure diffusion case.

Proposition 6. *The optimal coupon* C^* *that maximizes* $V_*(\tau_I; \tau_D^*, \theta^*, C^*)$ *in (17) has the form*

$C^* = \psi X(\tau_I)$, where ψ is the unique positive solution to the following equation

$$\frac{\phi}{r} = A_2\psi^{\beta_3} + A_3\psi^{\beta_4},$$

where

$$A_2 = \frac{\phi}{r}d_{1,0}(1+\beta_3)\gamma^{\beta_3} + \frac{\alpha(1-\phi)d_{1,1}(1+\beta_3)\gamma^{1+\beta_3}}{r-\mu^{\theta^*}},$$
$$A_3 = \frac{\phi}{r}d_{2,0}(1+\beta_4)\gamma^{\beta_4} + \frac{\alpha(1-\phi)d_{2,1}(1+\beta_4)\gamma^{1+\beta_4}}{r-\mu^{\theta^*}},$$

β_i, $d_{i,j}$ are given by (14), and γ satisfies $X_D^* = \gamma C$ with $\gamma = \frac{\beta_3\beta_4(\eta_2^*+1)}{(\beta_3+1)(\beta_4+1)\eta_2^*}\frac{r-\mu^{\theta^*}}{r}$ as in (15).

3. The Value of the Growth Option

Substituting the solutions to $E(\tau_I;\tau_D^*,\theta^*,C^*)$ and $D(\tau_I;\tau_D^*,\theta^*,C^*)$ into $V_*(\tau_I;\tau_D^*,\theta^*,C^*)$, we obtain the firm value at τ_I as

$$V_*(\tau_I;\tau_D^*,\theta^*,C^*) = A_4X(\tau_I) - I, \tag{18}$$

where

$$A_4 = A_0 + \frac{\phi}{r}\psi\left[1 - d_{1,0}(\gamma\psi)^{\beta_3} - d_{2,0}(\gamma\psi)^{\beta_4}\right] - \alpha A_0\gamma\psi\left[d_{1,1}(\gamma\psi)^{\beta_3} + d_{2,1}(\gamma\psi)^{\beta_4}\right]$$

for A_0 given in Proposition 2. Thus, the value of the growth option under optimal financing at time 0 is

$$V_*(0;\tau_I^*,\tau_D^*,\theta^*,C^*) = \sup_{\tau_I} \inf_{Q^\theta} \mathbb{E}_0^\theta\big[e^{-r\tau_I} V_*(\tau_I;\tau_D^*,\theta^*,C^*)\big]. \tag{19}$$

Proposition 7. *The density generator that gives the minimum expectation in (19) is* $\theta^* = (\theta_W^*, \theta_N^*) = (\kappa, 1 - e^{-M_1 u}\boldsymbol{1}_{u\geq 0} - \boldsymbol{1}_{u<0})$ *for all* $t \in [0, \tau_I^*]$*. Furthermore, for* $X(0) = x$*, the value of the growth option has the expression*

$$V_*(0;\tau_I^*,\tau_D^*,\theta^*,C^*) = A_4 X_I^*\Big[c_{1,1}\Big(\frac{X_I^*}{x}\Big)^{-\beta_1} + c_{2,1}\Big(\frac{X_I^*}{x}\Big)^{-\beta_2}\Big] - I\Big[c_{1,0}\Big(\frac{X_I^*}{x}\Big)^{-\beta_1} + c_{2,0}\Big(\frac{X_I^*}{x}\Big)^{-\beta_2}\Big],$$

where the optimal stopping time satisfies $\tau_I^* = \inf_t\{X(t) \geq X_I^*\}$*, and the investment boundary* X_I^* *satisfies*

$$A_4 X_I^* = \underbrace{\frac{\beta_1\beta_2}{(\beta_1-1)(\beta_2-1)}\frac{\eta_1^*-1}{\eta_1^*}}_{\text{option multiplier}>1} I. \tag{20}$$

In the formulation above, $c_{i,j}$*,* β_i*, and* η_i^* *are the same as in Proposition 2, and* A_4 *is from (18).*

In equation (20), A_4 represents the VM under optimal financing, which is higher than the VM under equity financing, A_0. The difference between these two multipliers reflects the net tax benefit of debt. The tax benefit (TB) and bankruptcy cost (BC) at τ_I^* can be expressed as:

$$TB = \frac{\phi C^*}{r}\left(1 - \mathbb{E}_{\tau_I^*}^{\theta^*}\left[e^{-r(\tau_D^*-\tau_I^*)}\right]\right) = \frac{\phi\psi X(\tau_I^*)}{r}\left[1 - d_{1,0}(\gamma\psi)^{\beta_3} - d_{2,0}(\gamma\psi)^{\beta_4}\right],$$

$$BC = \alpha A_0 \mathbb{E}^{\theta^*}_{\tau_I^*}\left[X(\tau_D^*)e^{-r(\tau_D^*-\tau_I)}\right] = \alpha A_0 \gamma\psi X(\tau_I^*)\left[d_{1,1}(\gamma\psi)^{\beta_3} + d_{2,1}(\gamma\psi)^{\beta_4}\right].$$

E. The Investment Acceleration Benefit of Debt

We use the analytical solutions to discuss the general implications of (optimal) debt financing relative to all-equity financing. Throughout the rest of the paper, we use the Arrow-Debreu (AD) price of investment, $\mathbb{E}_0^{\theta^*}[e^{-r\tau_I^j}]$ for $j \in \{e, *\}$, to capture the expected time until investment occurs. The AD price represents the time-0 value of a unit payoff made at the stopping time τ_I^j, with higher AD prices indicating shorter expected times to investment.

Remark 4. *The net tax benefit of debt lowers the investment threshold, but its dominant effect is to raise both the AD price of investment and the project value.*

Using the results from Propositions 2 and 7, we have

(21)
$$\begin{aligned}
\frac{X_I^e}{X_I^*} &= \frac{A_4}{A_0} \\
&= 1 + \frac{\phi(r-\mu^*)}{(1-\phi)r}\psi\left[1 - d_{1,0}(\gamma\psi)^{\beta_3} - d_{2,0}(\gamma\psi)^{\beta_4}\right] - \alpha\gamma\psi\left[d_{1,1}(\gamma\psi)^{\beta_3} + d_{2,1}(\gamma\psi)^{\beta_4}\right] \\
&= 1 + \frac{\phi(r-\mu^*)}{(1-\phi)r}\psi\left[d_{1,0}\beta_3(\gamma\psi)^{\beta_3} + d_{2,0}\beta_4(\gamma\psi)^{\beta_4}\right] + \alpha\gamma\psi\left[d_{1,1}\beta_3(\gamma\psi)^{\beta_3} + d_{2,1}\beta_4(\gamma\psi)^{\beta_4}\right] > 1,
\end{aligned}$$

(22)
$$\frac{AD_*}{AD_e} = \left(\frac{A_4}{A_0}\right)^{\beta_1}(1 + A_5), \quad A_5 = \frac{c_{2,0}\left[\left(\frac{x}{X_I^*}\right)^{\beta_2-\beta_1} - \left(\frac{x}{X_I^e}\right)^{\beta_2-\beta_1}\right]}{c_{1,0} + c_{2,0}\left(\frac{x}{X_I^e}\right)^{\beta_2-\beta_1}} > 0,$$

and

$$(23)\qquad \frac{V_*}{V_e} = \left(\frac{A_4}{A_0}\right)^{\beta_1}(1+A_6), \quad A_6 = \frac{\frac{c_{2,0}}{\beta_2-1}\left[\left(\frac{x}{X_I^*}\right)^{\beta_2-\beta_1} - \left(\frac{x}{X_I^e}\right)^{\beta_2-\beta_1}\right]}{\frac{c_{1,0}}{\beta_1-1} + \frac{c_{2,0}}{\beta_2-1}\left(\frac{x}{X_I^e}\right)^{\beta_2-\beta_1}} > 0.$$

The third equality in (21) follows from the first-order condition for ψ (see the proof of Proposition 6.). The second part of the above Remark follows from $\beta_1 > 1$ and $A_4 > A_0$, which imply that a one-percent reduction in the investment threshold leads to more than a one-percent increase in the AD price and project value under debt financing.[13] Hence, β_1 captures the expected first-passage time effect, determined by the sample-path properties of $X(t)$. The second (or third) line in (21) defines the zero-leverage equivalent cash flow level—the level at which an unlevered firm has the same value as its optimally levered counterpart when the latter's cash flow level is 1. Moreover, A_5 and A_6 are close to zero because $x < X_I^* < X_I^e$, $\beta_2 - \beta_1$ is large, and the coefficients involving $c_{1,0}$ and $c_{2,0}$ remain moderate under realistic parameter values. Collectively, these observations imply that the acceleration benefit of debt reflects an interaction between the *base* (A_4/A_0) and the *power* ($\beta_1 > 1$). The benefit is further an interaction between β_3, the primary factor for the *base*, and β_1, the passage-time factor.

Remark 5. *Ambiguity can amplify the investment acceleration benefit of debt.*

As discussed earlier, although ambiguity weakens the upside potential of $X(t)$ and reduces the *base*, its impact on γ is secondary (see Remark 3). Furthermore, the concavity of A_4 in ψ (Proposition 6) implies that higher ambiguity reduces A_4 at a diminishing rate. Likewise, A_0 decreases as ambiguity increases, which slows the reduction of the *base*. Meanwhile, ambiguity

[13]Note that $\beta_1 > 1$ follows from $G(0) - r = -r < 0$, $G(1) - r = \mu^{\theta^*} - r < 0$, and $G(\infty) - r = \infty$.

raises β_1, which can amplify $(A_4/A_0)^{\beta_1}$ for two reasons: the slowing decline of A_4/A_0 and the dominance of the power effect when $A_4/A_0 > 1$. Overall, ambiguity enhances the gains in the AD price and project value by amplifying the power effect.

Remark 6. *Under a fixed cash flow threshold policy, the net tax benefit of debt leads firms to undertake larger innovation projects than under all-equity financing; however, ambiguity erodes this size advantage at a diminishing rate.*

This follows from equations (9) and (20):

$$A_0\bar{X} = OM \times I_e^*, \quad A_4\bar{X} = OM \times I_*^*, \quad \frac{I_*^*}{I_e^*} = \frac{A_4}{A_0} > 1,$$

with A_4/A_0 decreasing in ambiguity at a diminishing rate.

III. Quantitative Analysis and Implications

A. Parameter Choices

[Insert Table 2 approximately here]

Table 2 presents our parameter choices. The parameters in the first row correspond to the deterministic part of the model. These parameter values are drawn from the traditional literature on irreversible investment and capital structure (Leland, 1998; Goldstein et al., 2001; Sundaresan and Wang, 2007; Chen and Kou, 2009; Sundaresan et al., 2015). To make the quantitative analysis more realistic, we estimate the parameters related to the stochastic part of our model $\{\sigma, \lambda, \eta_1, \eta_2, p\}$ using DJX options data on the Dow Jones Industrial Average (DJIA) index. Because these parameters are inferred from option prices, they are risk neutral by construction and therefore align with our specification the reference measure as risk neutral. Under the assumptions of the Gordon growth model, the cash flow and equity value processes share the same stochastic component; see Goldstein et al. (2001). This feature is also consistent with the equilibrium setting of Kou (2002), where the equilibrium asset price is related to cash flows through the utility-gradient pricing kernel.

The DJIA index broadly represents the U.S. economy and includes large, industry-leading firms. These firms are not only well-established but also play a significant role in innovation. Moreover, DJX options are European-style and can only be exercised at maturity. This feature simplifies the pricing and allows for more accurate estimation of the model parameters. By contrast, pricing American options requires more complex numerical algorithms and accounting for the early-exercise premium, which may introduce additional numerical or estimation errors.

We use the full historical series of standardized DJX option contracts from the Ivy DB OptionMetrics volatility-surface dataset. Our sample spans from the inception of these options on October 6, 1997, through August 31, 2023, which is the latest available date in the database. In addition, for each option on each trading date, we collect the corresponding closing index level, continuous dividend yield, and the term structure of risk-free interest rates provided by Ivy DB.[14] We focus on options with more than 30 days to expiration and moneyness in the range $S/K \in (0.95, 1.05)$, as these contracts tend to be more liquid. Moreover, since our model is based on an infinite horizon, incorporating longer-dated options is more appropriate for estimation.

We estimate the risk-neutral parameters $(\sigma, \lambda, p, \eta_1, \eta_2)$ using a standard approach from

[14] This dataset provides standardized options with fixed expiration dates and delta intervals. A key advantage is that the data provider conducts rigorous validation checks to ensure data accuracy and constructs prices using closing bid-ask midpoints. See the Internet Appendix for more details about the data.

the empirical option pricing literature (e.g., Bakshi, Cao, and Chen, 1997; Huang and Wu, 2004). Specifically, for each trading day, we obtain parameter estimates by minimizing the sum of squared errors between model-implied and market-observed put option prices.[15] The parameter values reported are time-series averages of these daily estimates. A two-sided t-test against the null hypothesis of zero mean rejects the null at the 1% significance level for all parameters.

Panel B of Table 2 reports the first four central moments of $Y(t)$ for $t = 1$.[16] The variance is 0.069, corresponding to a standard deviation of 0.26. Most of this variation arises from the jump component, as indicated by the relatively low diffusion volatility ($\sigma = 0.118$). The skewness and kurtosis are -0.824 and 4.93, respectively—values that are broadly consistent with those reported by Conrad, Dittmar, and Ghysels (2013), who analyze the full universe of OptionMetrics data. In their Table 1, the median skewness ranges from -1.34 to -0.30, and the median kurtosis lies between 3.68 and 7.70 over the 1996–2005 period.

B. Quantifying Ambiguity

We employ relative entropy and detection-error probabilities to gauge ambiguity by comparing the likelihoods implied by an alternative model, Q^θ, with those of the reference model, Q^0. When ambiguity is small, the alternative model is close to the reference model in terms of relative entropy growth, and thus it is statistically challenging to differentiate them based on a finite sample of trajectory observations. Conversely, when ambiguity is large, it is relatively easy

[15] The option pricing formulas for an underlying asset following a double-exponential jump-diffusion process are provided in the Internet Appendix. Estimation based on call option prices produces very similar results.

[16] Formulas for the first four central moments of $Y(t)$ are given in the Internet Appendix.

to distinguish them statistically. Thus, a plausible scope of ambiguity requires considering a set of alternative models that are statistically close to the reference model.

Formally, the relative entropy measures the distance between a pair of probability measures. Given Q^θ and Q^0, we can write the growth in entropy of Q^θ relative to Q^0 over the time interval $[t, t+\Delta t]$, $\mathcal{R}(\theta_t)$, as

$$G(t, t+\Delta t) = \mathbb{E}_t^\theta\Big[\ln\Big(\frac{Z^\theta(t+\Delta t)}{Z^\theta(t)}\Big)\Big], \quad \mathcal{R}(\theta_t) = \lim_{\Delta t \to 0} \frac{G(t, t+\Delta t)}{\Delta t} \quad t \geq 0.$$

We denote by h the relative entropy growth under the worst-case measure, i.e., $h = \mathcal{R}(\theta^*)$. The independence of the diffusion and jump components (*per* Itô-Lévy Decomposition) implies that diffusion ambiguity and jump ambiguity contribute additively to relative entropy growth, i.e., $\mathcal{R}(\theta^*) = \mathcal{R}(\theta_W^*) + \mathcal{R}(\theta_N^*)$ and $h_W + h_N = h$. We use h_W and h_N to regulate the respective contributions of diffusion ambiguity and jump ambiguity to total relative entropy.

For the double exponential jump-diffusion process, we derive the relative entropy growth $\mathcal{R}(\theta^*)$ as (see the Internet Appendix for the proof)

$$\mathcal{R}(\theta^*) = \frac{1}{2}\kappa^2 + \Big(\frac{1}{\eta_1} - \frac{2}{\eta_1^*} + \frac{1}{\eta_1^* + M_1}\Big)\lambda p \eta_1 - \Big(\frac{M_1}{(\eta_1^*)^2} - \frac{1}{\eta_1^*} + \frac{1}{\eta_1^* + M_1}\Big)\lambda^* p^* \eta_1^*,$$

where η_1^*, p^*, and λ^* are the same as in Proposition 2. Moreover, given that $h_W + h_N = h$, we have

$$h_W = \frac{1}{2}\kappa^2, \text{and } h_N = \Big(\frac{1}{\eta_1} - \frac{2}{\eta_1^*} + \frac{1}{\eta_1^* + M_1}\Big)\lambda p \eta_1 - \Big(\frac{M_1}{(\eta_1^*)^2} - \frac{1}{\eta_1^*} + \frac{1}{\eta_1^* + M_1}\Big)\lambda^* p^* \eta_1^*,$$

by which we can recover values of κ and M_1 based on h and h_N (or h_W).

The detection-error probability quantifies relative entropy growth (h) that seems plausible to the decision maker. Let $\zeta^{\theta}(t) = \ln\big(Z^{\theta}(t)\big)$ denote the log of the Radon-Nikodym derivative process. The detection-error probability is defined as (assuming an equal prior on Q^{θ} and Q^{0})

$$\pi(t,T;h) = \frac{1}{2}\Big[Q^{0}\Big\{\zeta^{\theta}(T) > 0\Big|\mathcal{F}_t\Big\} + Q^{\theta}\Big\{\zeta^{\theta}(T) < 0\Big|\mathcal{F}_t\Big\}\Big]. \tag{24}$$

The first term within the right-hand side bracket represents the probability of the agent mistakenly choosing model Q^{θ} over the reference model Q^{0}, given a history of length $T-t$ generated from the state process under Q^{0}, while the second term is the probability of the agent erroneously favoring model Q^{0} over model Q^{θ}, given a history of length $T-t$ produced from the state process under Q^{θ}. In our analysis, we seek to identify the maximum level of ambiguity, h^{*}, such that $\pi(t,T;h^{*}) = 5\%$. This corresponds to the decision maker using a 5% confidence level when rejecting implausible models. Anderson et al. (2003) adopt a detection-error probability of 10%, while Bidder and Smith (2012) and Croce, Nguyen, and Schmid (2012) use lower values at 2.5% and 1.15%, respectively.

Aït-Sahalia and Matthys (2019) provide a method for calculating the detection-error probability in the presence of jump ambiguity, based on the conditional Fourier transform of $\zeta^{\theta^{*}}(t)$. Given a sample of length n, the detection-error probability is

$$\begin{aligned}\pi(t,n;h) &= \frac{1}{2}\Big[Q^{0}\Big\{\zeta^{\theta^{*}}(n) > 0\Big|\mathcal{F}_t\Big\} + Q^{\theta^{*}}\Big\{\zeta^{\theta^{*}}(n) < 0\Big|\mathcal{F}_t\Big\}\Big], \quad t \geq 0,\ n = mT \\ &= \frac{1}{2} - \frac{1}{2\pi}\int_{\mathbb{R}+}\Big(\Re\Big[\frac{\hat{\zeta}^{\theta^{*}}_{\theta^{*}}(u,t,n)}{iu}\Big] - \Re\Big[\frac{\hat{\zeta}^{\theta^{*}}_{0}(u,t,n)}{iu}\Big]\Big)du,\end{aligned} \tag{25}$$

where T is the number of years, m is the sampling frequency (given that we use annualized parameter values), $i \equiv \sqrt{-1}$, $\Re(\cdot)$ denotes the real part of a complex number, and the two conditional Fourier transforms are

$$\hat{\zeta}_0^{\theta^*}(u,t,n) = \mathbb{E}_t\left[e^{iu\zeta^{\theta^*}(n)}\right], \quad \text{and} \quad \hat{\zeta}_{\theta^*}^{\theta^*}(u,t,n) = \mathbb{E}_t^{\theta^*}\left[e^{iu\zeta^{\theta^*}(n)}\right].$$

We explicitly calculate the two conditional Fourier transforms, $\hat{\zeta}_0^{\theta^*}(u,t,n)$ and $\hat{\zeta}_{\theta^*}^{\theta^*}(u,t,n)$, in the Internet Appendix.

To implement the calibration, we need to specify the ratio h_W/h and set $n = mT$. In our baseline analysis, we use $h_W/h = 0.5$, implying that the decision maker places equal weight on jump ambiguity and drift ambiguity.[17] For the sample length used to calculate detection error probabilities, we assume access to thirty years of data ($T = 30$) to gauge the extent of sample length in determining the ambiguity level allowed for the EBIT dynamics. We choose this horizon because our focus is on growth options rather than assets in place, and extending the sample length beyond 30 years would draw on more distant periods that are less informative for our analysis. Additionally, we select a quarterly frequency ($m = 4$), which is the highest frequency available from Compustat.

[Insert Figure 2 approximately here]

Figure 2 presents the calibration results for detection-error probabilities. Panel (a) shows that as ambiguity (h) increases, $\pi(n;h)$ decreases, indicating that it becomes easier to distinguish the worst-case model from the reference model. For our benchmark parameters, with $h_W/h = 0.5$, the critical value h^* satisfying the lower bound $\pi(n;h^*) = 0.05$ is 0.049. Panels (b) and (c) illustrate how the the ambiguity parameters κ and M_1 vary with relative entropy growth.

[17] To the best of our knowledge, no study has attempted to empirically estimate the relative contributions of each type of ambiguity. We therefore assume equal weights for each type.

With $h_W/h = 0.5$, the maximum drift ambiguity (κ^*) is about 0.221, implying a drift reduction of $\kappa^*\sigma = 0.221 \times 0.118 = 0.026$ from 0.02 under the reference measure. On the other hand, the maximum jump ambiguity (M_1^*) is significant, around 4.142, lowering the conditional mean log positive jump size from 1/8.343 to about 1/12.485 (Panel (d)). Additionally, Panel (e) shows that the jump intensity (λ^*) decreases from approximately 1.986 to 1.912. This result is consistent with our discussion at the end of Section II.B and after Proposition 2 in Section II.C, where we show that λ^* attains an interior value because the worst-case distortion minimizes the unconditional probability of positive jumps while maximizing that of negative jumps. Lastly, Panel (f) shows that the conditional probability of positive jumps (p^*) drops to less than 0.078 from 0.112. Collectively, jump ambiguity minimizes the influence of positive jumps while amplifying that of negative jumps.

Figure 3 presents the first four central moments of $Y(t)$ as a function of relative entropy growth (h), demonstrating how jump ambiguity affects the distribution of $Y(t)$ in a way that is distinct from drift ambiguity. As relative entropy growth rises, the variance of $Y(t)$ decreases and skewness becomes more negative. Jump ambiguity thins the right tail of the distribution while thickening the left tail, and as a result, higher levels of ambiguity increase kurtosis.

[Insert Figure 3 approximately here]

Our model features a path-dependent option whose value hinges on the sample-path characteristics of the underlying process. In particular, the first-passage time distribution—which captures the probability that the process reaches a given threshold for the first time—is the key determinant. In our model, ambiguity about jump risk shifts the entire distribution leftward and amplifies its asymmetry. To understand its implications for path-dependent decisions, we analyze how jump ambiguity affects the first-passage time probabilities of reaching levels above and below the current state, as shown in Figure 4.

[Insert Figure 4 approximately here]

Figure 4 illustrates how ambiguity in the return distribution affects first-passage dynamics. Specifically, ambiguity reduces the expected first-passage time to an upper level when approaching from below (Panel (a)) and increases it when approaching a lower level from above (Panel (b)). The results clearly indicate that as ambiguity h increases, AD_u drops sharply while AD_d rises substantially. These patterns form the basis for interpreting the subsequent results.

C. Investment and Financing Decisions

[Insert Figure 5 approximately here]

We next examine the impact of ambiguity on investment and financing decisions. Consistent with Remark 1, which shows that ambiguity reduces the investment payoff at τ_I^e and prolongs the expected time to investment, Figure 5 demonstrates that ambiguity lowers both the project value and the AD price of investment while raising the investment threshold. The resulting decline in project value suggests that investors demand substantial compensation for bearing ambiguity. Notably, the effects of ambiguity stand in stark contrast to those of volatility, which typically boost project value. The difference arises because the ambiguity-averse entrepreneur bases her decision on the least favorable EBIT profile.

The results in Figure 5 also indicate the benefits of debt in terms of the investment boundary and project value, in line with the theoretical analysis in Section II.E. Figure 6 further illustrates the mechanism underlying the benefits of debt. In the absence of ambiguity, we find that $X_I^e/X_I^* = 1.063$, $AD_*/AD_e = 1.094$, and $V_*/V_e = 1.094$, which correspond to $\beta_1 = 1.468$.

Additionally, the values of A_6 on the order of 10^{-8}, shown in Panel (e), are consistent with our earlier discussion following Remark 4. Without ambiguity, the amplification effect through the power coefficient β_1 is small, and thus the benefit of debt financing in terms of enhancing the project value is very limited.

Panel (a) shows that the ratio X_I^e/X_I^* decreases with ambiguity at a diminishing rate, consistent with the concavity of the *base* (equation (21)) in ambiguity. This concavity can be further explained by the concave relationship between β_3 and ambiguity. More importantly, Panels (b) and (c) highlight the dominance of the power effect. Although the base value in Panel (a) declines modestly from 1.063 to 1.048 as ambiguity increases, the power coefficient β_1 rises substantially from 1.468 to 2.687, as shown in Panel (d). The substantial effect of ambiguity on β_1 drives the overall increase in both AD_*/AD_e and V_*/V_e. Notably, the change in V_*/V_e by varying the ambiguity level is quantitatively meaningful: without ambiguity, $V_*/V_e - 1 = 9.42\%$, while at the highest attainable ambiguity level h^*, $V_*/V_e - 1 = 13.48\%$—representing a 43% increase. This result suggests that, in contrast to the scenario without ambiguity, debt financing greatly enhances the project value under ambiguity.

[Insert Figure 6 approximately here]

Figure 7 shows the optimal capital structure. Because the default boundary (X_D^*), coupon (C^*), levered equity, debt, and net tax benefit depend on the realized cash flow at investment ($X(\tau_I^*)$), we scale them by $X(\tau_I^*)$ to avoid the "overshooting issue" ($X(\tau_I^*) \geq X_I^*$). The AD price of default, leverage, and cost of debt ($C^*/D(\tau_I^*)$) are already ratios and need no adjustment. These normalized quantities resemble those in the standard assets-in-place financing problem, where the cash flow at financing is exogenously fixed (e.g., Goldstein et al., 2001). In our growth-option setting, by contrast, the cash flow at investment ($X(\tau_I^*)$) is endogenously determined and varies with model parameters.

[Insert Figure 7 approximately here]

The results in Figure 7 are consistent with our earlier discussion on the effects of ambiguity and the behavior of the cash flow process under the worst-case scenario, which are elaborated in the remarks in Section D. As shown previously in Figure 4, ambiguity reduces the expected first-passage time to a fixed level from above (τ_d). This helps explain the decline in the default threshold X_D^* and the increase in the AD price of default observed in Figure 7. Specifically, the optimal default boundary drops moderately from 0.275 to 0.209 as ambiguity increases from zero to its maximum, in line with Remark 3.

Moreover, as the optimal coupon C^* declines with ambiguity, the values of levered equity, debt, leverage, and the net tax benefit (NTB) of debt all decrease. This result contrasts sharply with the effect of volatility, which tends to increase the levered equity value while reducing the debt value. The cost of debt rises only slightly—from 6.25% to 6.97%—since both C^* and the debt value decline, with the latter falling at a marginally faster rate. These numerical results on the cost of debt are reasonable for an investment-grade firm, given that our model is calibrated using DJX option data.

An additional insight comes from the NTB. Without ambiguity $h = 0$, the NTB equals 1.687, and the total firm value is $A_4 = A_0 + NTB = (1 - \phi)/(r - \mu) + NTB = 28.354$, implying that the NTB accounts for about 6% of total firm value. In the presence of ambiguity, $h = h^*$, this proportion drops to 4.6% moderately. The range of the NTB as a fraction of firm value implied by our model is broadly consistent with the empirical estimates reported by Korteweg (2010).

Summary of Comparative Statics.

We conduct additional comparative statics to examine how the benefits of debt and the optimal capital structure respond to key model features. Detailed results are presented in the Internet Appendix. In the baseline case, drift and jump ambiguity contribute equally to relative entropy growth. We then vary the relative share of jump (or drift) ambiguity to assess its impact on the equilibrium outcome and, in a separate exercise, study how the sample length used to calibrate ambiguity affects the results. In both analyses, all other parameters remain at their benchmark values.

Our first analysis on the ambiguity share reveals a key result: the relevance of drift versus jump ambiguity is context dependent, differing between the optimal financing of a growth option and that of an asset in place. The ambiguity share directly influences the expected first-passage time to reach a given level from above or below. Financing an asset in place concerns only the passage from above, i.e., the default threshold, while financing a growth option involves both entry and default thresholds. Hence, the ambiguity share that leads to the highest optimal leverage does not necessarily lead to the highest investment acceleration effect of debt, as the underlying mechanism differs across contexts. Consequently, the debt level that maximizes the value of an asset in place does not necessarily maximize the value of a growth option. This distinction implies that the impact of ambiguity depends on the project's nature: ambiguity about jumps plays a more significant role for innovation projects (Kerr and Nanda, 2015), while ambiguity about the drift matters more for assets in place. Accordingly, financing policies should be tailored to the type of project, as ambiguity may alter investment and financing thresholds differently across contexts.

Our second analysis reveals that debt financing is especially valuable for innovative projects, which are characterized by high levels of ambiguity. A shorter sample used to calibrate ambiguity reflects limited historical information, reducing the ability to distinguish the worst-case measure from the reference measure and thereby increasing perceived ambiguity. Quantitatively, halving the sample size nearly doubles total ambiguity and amplifies the gains from debt. Intuitively, projects with fewer comparable precedents—such as new technologies or young firms—face greater uncertainty about future cash flows. Our results highlight that the investment acceleration benefits of debt are therefore stronger for such projects.

D. Optimal Project Size

To examine how ambiguity affects the optimal project size under the two financing scenarios, we assume that the firm follows a fixed cash flow threshold policy, as discussed in Remarks 2 and 6. Specifically, we set the cash flow threshold at $\bar{X} = 2X(0) = 2x$, so that investment occurs when the cash flow level doubles its initial value. Given this threshold, the firm chooses the optimal project size I_i^* (for $i \in \{e, *\}$) to maximize firm value, where $I_i^* = I\bar{X}/X_I^i$. Figure 8 plots the optimal size as a function of relative entropy growth $h \in [0, h^*]$, in our baseline setting where ambiguity is equally shared across the two types.

[Insert Figure 8 approximately here]

Overall, the results are consistent with Remarks 2 and 6. Panel (a) shows that in the absence of ambiguity, the optimal project size under equity financing is nearly halved relative to the baseline level ($I = 100$), whereas optimal financing supports a larger project, with an optimal size close to 56. Since the value function is linear in project size under both financing scenarios, this increase in size under optimal financing yields a proportional gain in project value. The net tax benefit of debt further amplifies this gain. Thus, the larger size under optimal financing results in a magnified improvement in firm value. As ambiguity rises, the optimal size declines under both financing modes. Panel (b) plots the ratio I_*^*/I_e^*, which mirrors the pattern of the ratio $X_I^e/X_I^* = A_4/A_0$ in Figure 6, in line with Remark 6. These findings suggest that firms respond to ambiguity by downsizing their projects, while optimal financing mitigates the adverse effect.

IV. Conclusion

We have examined a real options model in which an ambiguity-averse entrepreneur considers investing in an innovation project and financing the investment optimally. The novel features of our model are the introduction of jumps in innovation returns and ambiguity about the EBIT dynamics characterized by the double exponential jump-diffusion model developed by Kou (2002). We employ a set of priors to characterize both drift ambiguity and jump ambiguity within the multiple priors utility framework and solve the model analytically. Our results show that debt accelerates investment because tax benefits lower the investment threshold and amplify the passage time effect, raising project value. Ambiguity increases thresholds and reduces values under both financing modes, but optimal financing mitigates these effects. Hence, the investment acceleration benefit of debt becomes stronger when ambiguity is greater.

Consistent with a large literature highlighting the positive effects of debt on innovation (e.g., Benfratello et al., 2008; Amore et al., 2013; Chava et al., 2013), our theoretical results demonstrate that innovation projects—characterized by both jump and drift ambiguity—benefit substantially from debt financing. Our comparative statics results on the sample length used for calibrating ambiguity help explain the empirical results that young startups rely heavily on debt financing and that better access to debt financing leads young and private firms to increase the rate and novelty of their innovations (Robb and Robinson, 2014; Davis et al., 2020; Chava et al., 2013). Moreover, our result on the impact of ambiguity on the optimal project size aligns with the empirical insight of Campello et al. (2024), who show that sunk costs of investment (denoted by I in our model) shape the sensitivity of firm behavior to uncertainty. Finally, we show that optimal leverage is higher when jump ambiguity is the dominant concern, in line with the empirical finding of Robb and Robinson (2014), based on the Kauffman Firm Survey, that debt plays a dominant role for startups' financing choices.

Future research could extend our analysis in several directions. One avenue would be to incorporate the timing differences in cash flows between innovation and fixed-asset investment, following Campello and Kankanhalli (2024) who emphasize two key features of innovation: its "race to patent" nature, and the staged nature of R&D investment. Another direction would be to introduce external financing constraints or asymmetric information among agents to explore how these frictions interact with ambiguity in shaping investment and financing decisions. Future empirical work could investigate whether firms facing greater ambiguity, especially those engaged in innovative or intangible-intensive projects, use debt financing to mitigate the adverse effects of uncertainty on investment scale.

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TABLE 1

Characteristics of the Lévy Measure under Multiple Priors

This table presents the characteristics of the Lévy measure under the reference measure Q^0, alternative equivalent priors Q^θ for $\theta_{N,1}(t) \in [-M_1, 0]$ and $\theta_{N,2}(t) \in [0, M_2]$ and the worst-case measure Q^{θ^*}.

Quantity	Q^0	Q^θ	Q^{θ^*}
Jump intensity	λ	$\lambda_t^\theta \in \left[\lambda\left(\frac{p\eta_1}{\eta_1+M_1} + \frac{q\eta_2}{\eta_2+M_2}\right), \lambda\right]$	$\lambda^* = \lambda\left(\frac{p\eta_1}{\eta_1+M_1} + q\right)$
Cond. Prob. (positive jumps)	p	$p_t^\theta \in \left[\frac{p\eta_1}{p\eta_1+q(\eta_1+M_1)}, \frac{p(\eta_2+M_2)}{p(\eta_2+M_2)+q\eta_2}\right]$	$p^* = \frac{p\eta_1}{p\eta_1+q(\eta_1+M_1)}$
Cond. Prob. (negative jumps)	$q = 1-p$	$q_t^\theta \in \left[\frac{q\eta_2}{p(\eta_2+M_2)+q\eta_2}, \frac{q(\eta_1+M_1)}{p\eta_1+q(\eta_1+M_1)}\right]$	$q^* = \frac{q(\eta_1+M_1)}{p\eta_1+q(\eta_1+M_1)}$
Cond. mean jump size (positive)	$\frac{1}{\eta_1}$	$\int_{\mathbb{R}^+} u f_t^\theta(du) \in \left[\frac{1}{\eta_1+M_1}, \frac{1}{\eta_1}\right]$	$\frac{1}{\eta_1+M_1}$
Cond. mean jump size (negative)	$\frac{-1}{\eta_2}$	$\int_{\mathbb{R}^-} u f_t^\theta(du) \in \left[\frac{-1}{\eta_2}, \frac{-1}{\eta_2+M_2}\right]$	$-\frac{1}{\eta_2}$

TABLE 2

Benchmark Parameter Values

Panel A reports the benchmark parameter values. α is the liquidation cost. ϕ is the tax rate. r is the risk-free rate. I is the cost of the project. x is the initial EBIT value. The remaining parameters govern the EBTI dynamics equation (1). We add a hat to those parameters estimated from the DJX options data. Panel B reports the central moments of $Y(t) = \ln(X(t)/X(0))$, under the reference measure for $t = 1$.

A. Parameter Values					
$r = 0.05$	$\phi = 0.2$	$\alpha = 0.5$	$I = 100$	$x = 3$	$\mu = 0.02$
$\hat{\sigma} = 0.118$	$\hat{\lambda} = 1.986$	$\hat{\eta_1} = 8.343$	$\hat{\eta_2} = 8.482$	$\hat{p} = 0.112$	
B. Central Moments of $Y(t) = \ln(X(t)/X(0))$ under Q^0 for $t = 1$					
Mean $= -0.013$	Variance $= 0.069$	Skewness $= -0.824$	Kurtosis $= 4.93$		

FIGURE 1

Timeline

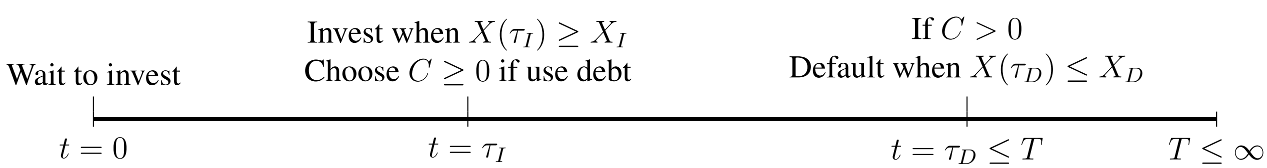
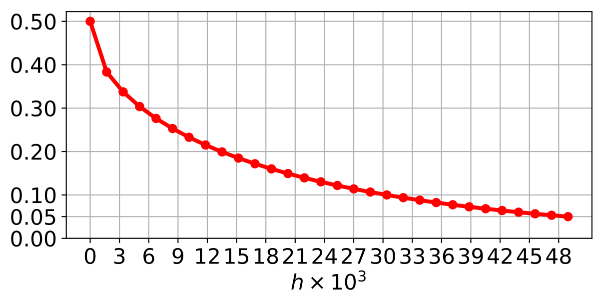


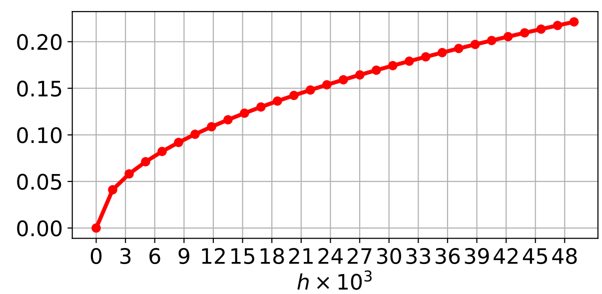
FIGURE 2

Calibration of Ambiguity

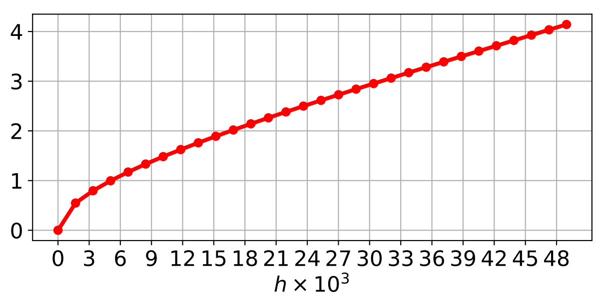
This figure plots the detection-error probability ($\pi(n; h)$), the parameters governing ambiguity (κ^* and M_1^*), and the parameters of the double exponential jump-diffusion process under both the worst-case measure (Q^{θ^*}) and the reference measure (Q^0) across different levels of ambiguity. h denotes the total relative entropy growth bound. The maximum h^* is set such that $\pi(n; h^*) = 0.05$. The sample length for calculating detection-error probabilities is $n = 120$ (quarters), and we set $h_W/h = 0.5$.



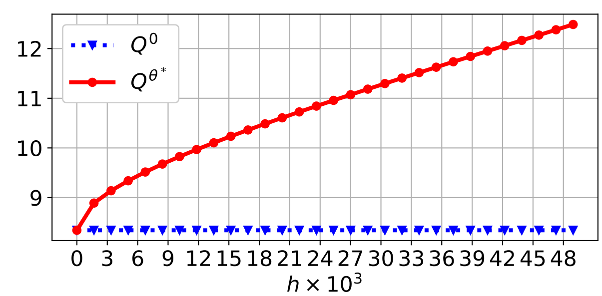
(a) $\pi(n; h)$



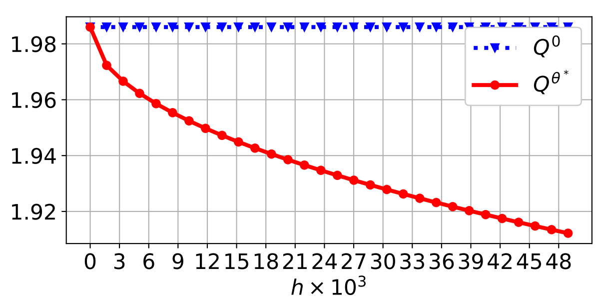
(b) κ^*



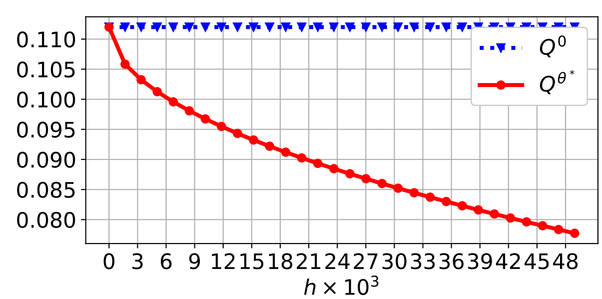
(c) M_1^*



(d) η_1^*



(e) λ^*

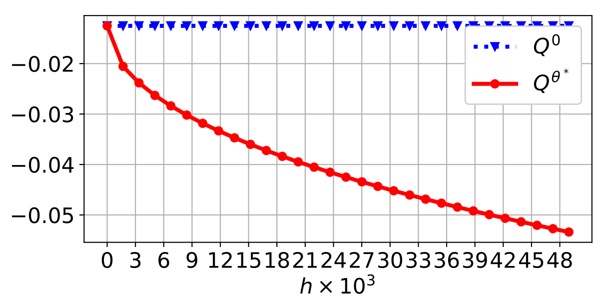


(f) p^*

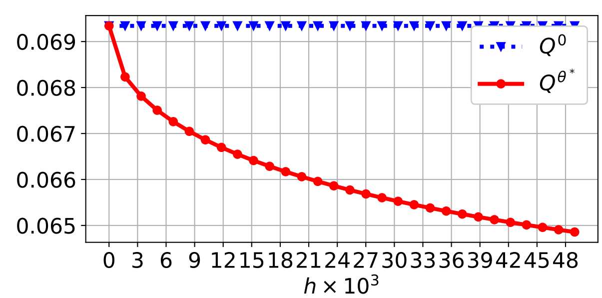
FIGURE 3

Central Moments under the Worst-Case Measure

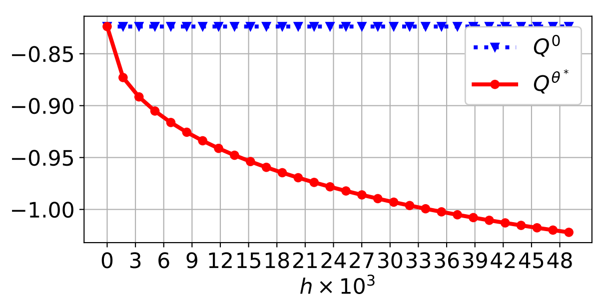
This figure plots the first four central moments of $Y(t)$ based on the calibrated parameters plotted in Figure 2. h denotes the total relative entropy growth bound. The maximum h^* is set such that $\pi(n; h^*) = 0.05$. The sample length for calibrating ambiguity is $n = 120$ (quarters), and we set $h_W/h = 0.5$.



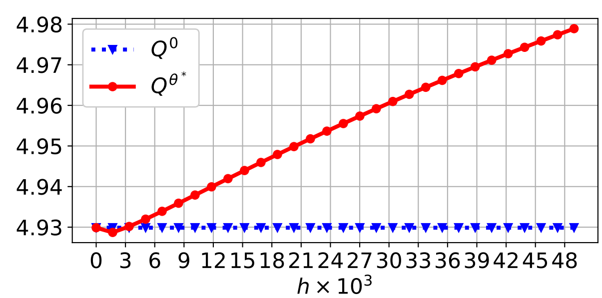
(a) Mean



(b) Variance



(c) Skewness

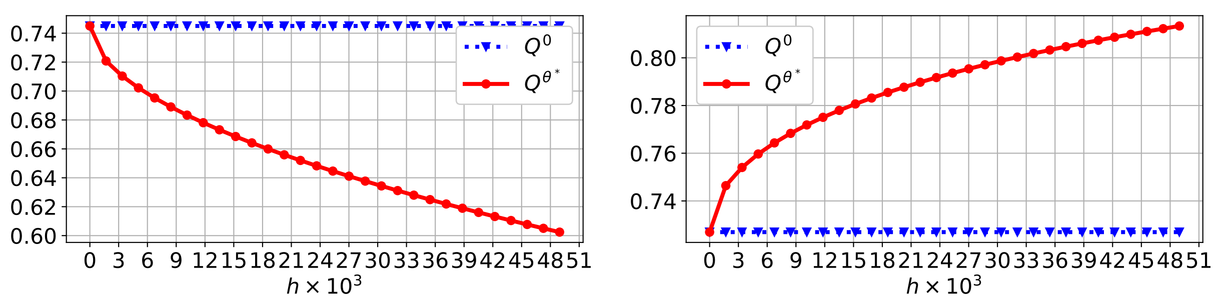


(d) Kurtosis

FIGURE 4

First-Passage Time under the Worst-Case Measure

This figure plots the AD price of reaching a fixed upper level from below $AD_u := \mathbb{E}[e^{-r\tau_u}]$ for $\tau_u := \inf\{t \geq 0; X(t) \geq u\}$ and the AD price of reaching a fixed lower level from above $AD_d := \mathbb{E}[e^{-r\tau_d}]$ for $\tau_d := \inf\{t \geq 0; X(t) \leq d\}$ across different ambiguity levels. We set $u = 1.2x$ and $d = 0.8x$. h denotes the total relative entropy growth bound. The maximum h^* is set such that $\pi(n; h^*) = 0.05$. The sample length for calibrating ambiguity is $n = 120$ (quarters), and we set $h_W/h = 0.5$.



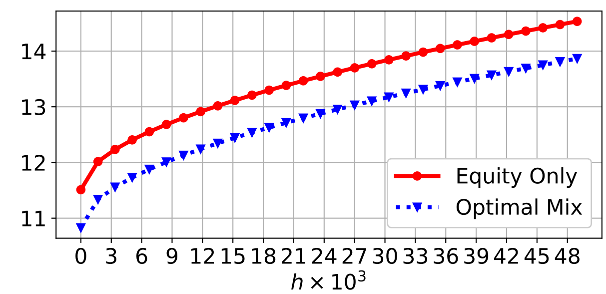
(a) AD_u

(b) AD_d

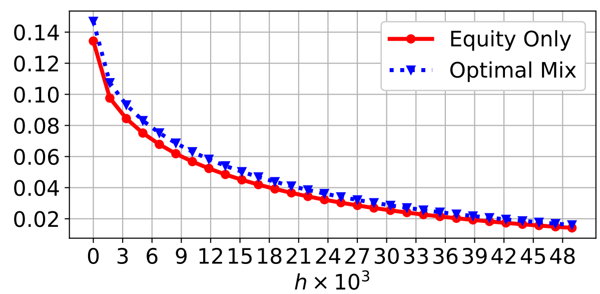
FIGURE 5

The Value of Investment

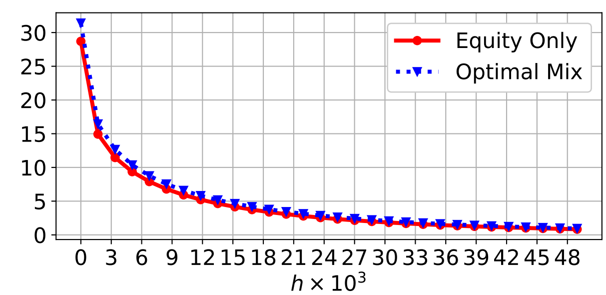
Panels (a) to (c) plot the the optimal investment boundary X_I^j, the AD price of investment AD_I^j, and project value $V_j(0)$ against relative entropy growth, where j denotes equity financing (e) or optimal financing $(*)$. h denotes the total relative entropy growth bound. The maximum h^* is set such that $\pi(n; h^*) = 0.05$. The sample length for calibrating ambiguity is $n = 120$ (quarters). We set $h_W/h = 0.5$.



(a) Investment Boundary



(b) AD Price (Invest)

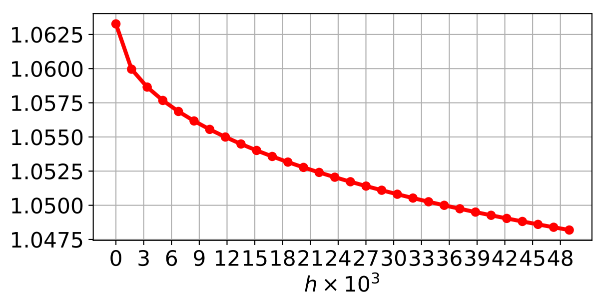


(c) Project value

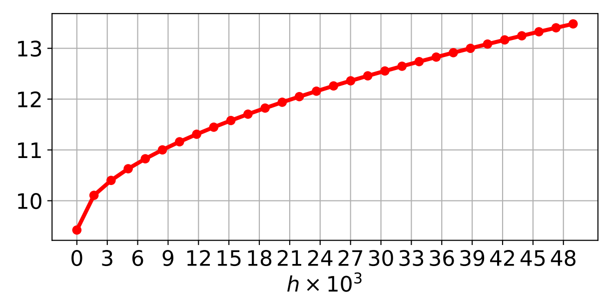
FIGURE 6

The Value-Enhancing Effect of Debt

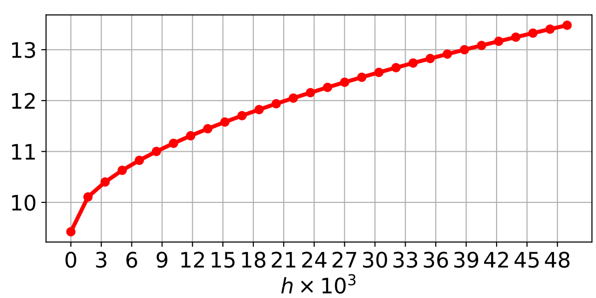
Panels (a)—(c) plot X_I^e/X_I^*, AD_*/AD_e, and V_*/V_e against relative entropy growth h. The expressions of X_I^e/X_I^*, AD_*/AD_e, and V_*/V_e are given in equations (21), (22), and (23). Panel (d) plots β_1, the smaller positive root of $G(\beta) = r$, against h. Panel (e) plots the coefficient A_6 in (23). Panel (f) plots β_3, the larger negative root of $G(\beta) = r$, against h. The maximum h^* is set such that $\pi(n; h^*) = 0.05$. The sample length used for calibrating ambiguity is $n = 120$ (quarters) and we set $h_W/h = 0.5$.



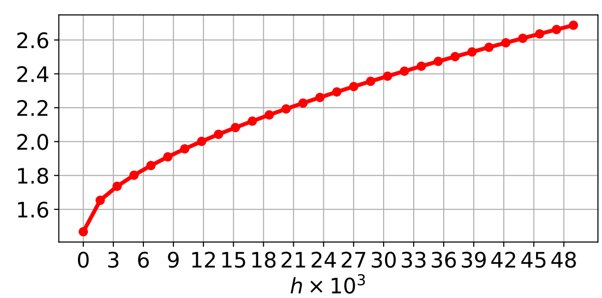
(a) X_I^e/X_I^*



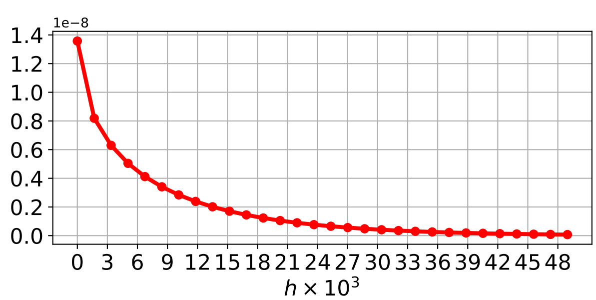
(b) $AD^*/AD^e - 1$ (%)



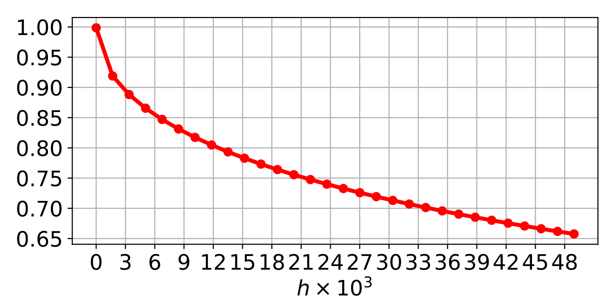
(c) $V_*(0)/V_e(0) - 1$ (%)



(d) β_1



(e) A_6

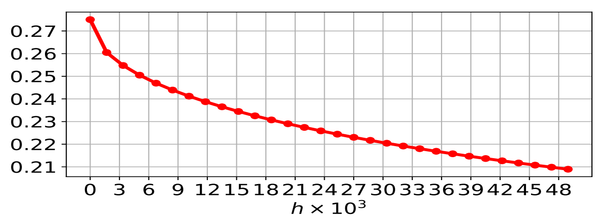


(f) β_3

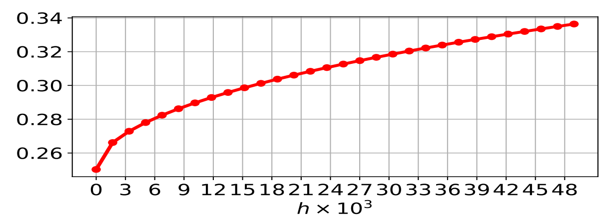
FIGURE 7

Optimal Capital Structure

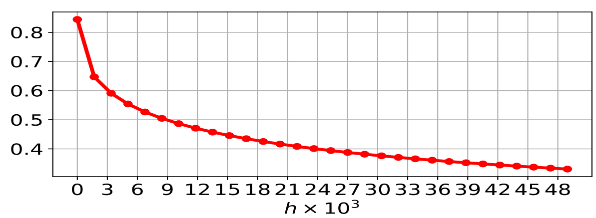
This figure plots the the scaled optimal default boundary X_D^*, the AD price of default, the scaled optimal coupon C^*, the scaled levered equity and debt, optimal leverage $D/(D+E)$, the scaled net tax benefit, and the cost of debt $(C^*/D(\tau_I^*))$ against ambiguity. The scaled quantities are divided by $X(\tau_I^*)$. The maximum h^* is set such that $\pi(n;h^*) = 0.05$. The sample length used for calibrating ambiguity is $n = 120$ (quarters), and we set $h_W/h = 0.5$.



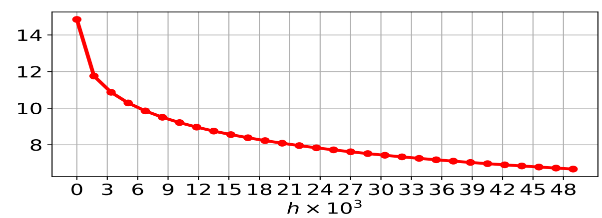
(a) X_D^*



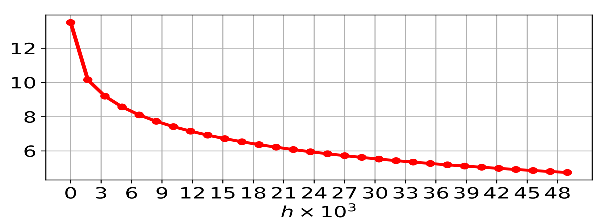
(b) *AD* Price (Default)



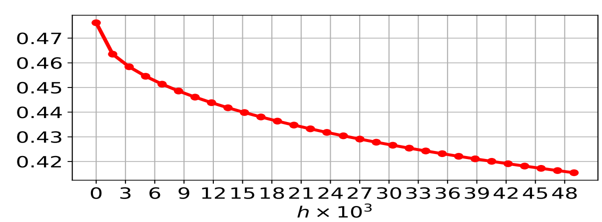
(c) C^*



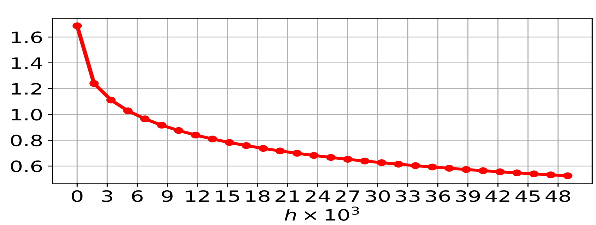
(d) Levered Equity



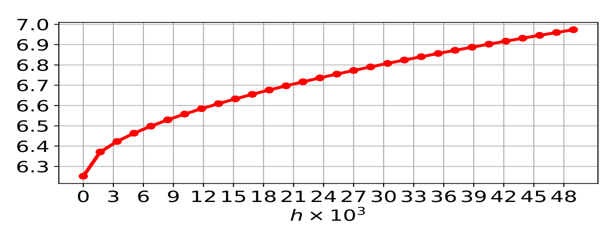
(e) Debt



(f) Leverage



(g) Net Tax Benefit

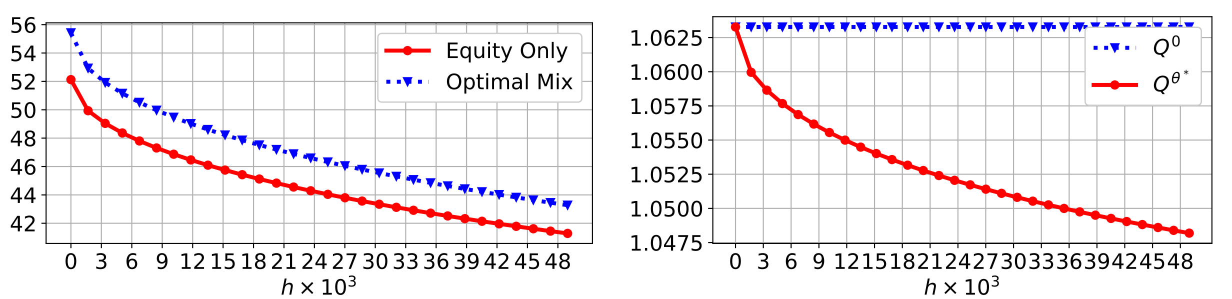


(h) Cost of Debt

FIGURE 8

The Optimal Project Size Choices under Ambiguity

This figure plots the optimal project size I_i^* for $i \in \{e, *\}$ under a fixed cash flow threshold policy. The fixed cash flow trigger $\bar{X} = 2X(0) = 2x$. The upper bound of h is set such that the detection-error probability is 5% with $n = 120$ (quarters) and $h_W/h = 0.5$.



(a) I_i^* (b) I_*^*/I_e^*

Internet Appendix

for

Financing Innovation under Ambiguity

IA.I. Proofs

Proposition 1

Proof. We can express (8) as

$$V_e(0;\tau_I,\theta^*) = \inf_{Q^\theta}\ \mathbb{E}_0^\theta\Big[\int_{\tau_I}^T e^{-rt}(1-\phi)X(t)dt - e^{-r\tau}I\Big], \quad \text{for any } \tau_I.$$

Because the expectation in the expression above is dynamically consistent, given our choice of Θ, we can start with

$$V_e(0;\tau_I,\theta^*) = \inf_{Q^\theta}\ \mathbb{E}_0^\theta\Big[\inf_{Q^\theta}\ \mathbb{E}_{\tau_I}^\theta\Big[\int_{\tau_I}^T e^{-rt}C_0X(t)dt\Big] - e^{-r\tau_I}\delta I\Big], \quad \text{for any } \tau_I,$$

or

$$V_1(\tau_I;T,\theta^*) = \inf_{Q^\theta} V_1(\tau_I;T,\theta) = \inf_{Q^\theta}\ \mathbb{E}_{\tau_I}^\theta\Big[\int_{\tau_I}^T e^{-rt}C_0X(t)dt\Big], \tag{IA.I.1}$$

where we use $C_0 = 1-\phi$ to simplify the expressions.

Under Q^θ, $\big(V_1(t;T,\theta),\Sigma_1(t;\theta),K_1(t,u;\theta)\big)$ is the solution to the following BSDE

$$-dV_1(t;T,\theta) = e^{-rt}C_0X(t)dt - \Sigma_1(t;\theta)dW^\theta(t) - \int_{\mathbb{R}} K_1(t,u;\theta)\tilde{N}^\theta(dt,du),$$

for $t \in [\tau_I, T]$ and $V_1(T;T,\theta) = 0$. By Girsanov's Theorem, we have under Q^0

(IA.I.2)

$$-dV_1(t;T,\theta) = \Big(e^{-rt}C_0X(t) - \Sigma_1(t;\theta)\theta_W(t) - \int_{\mathbb{R}} K_1(t,u;\theta)\theta_N(t,u)\nu(du)\Big)dt$$
$$- \Sigma_1(t;\theta)dW(t) - \int_{\mathbb{R}} K_1(t,u;\theta)\tilde{N}(dt,du),\ t \in [\tau, T],\ \text{and } V_1(T;T,\theta) = 0.$$

Meanwhile, by Itô formula, we have for $f^\theta(t, X(t)) = V_1(t;T,\theta)$

$$dV_1(t;T,\theta) = f^\theta_t dt + f^\theta_x\big(\mu^\theta X(t)dt + \sigma X(t)dW^\theta(t)\big) + \frac{1}{2}\sigma^2X^2(t)f^\theta_{xx}dt$$
$$+ \int_{\mathbb{R}} \Big(f^\theta\big(t, X(t^-) + (e^u-1)X(t^-)\big) - f^\theta(t, X(t^-)) - f^\theta_x\big(t, X(t^-)\big)(e^u-1)X(t^-)\Big)\nu^\theta(du)dt$$
$$+ \int_{\mathbb{R}} \Big(f^\theta\big(t, X(t^-) + (e^u-1)X(t^-)\big) - f^\theta\big(t, X(t^-)\big)\Big)\tilde{N}^\theta(dt,du).$$

Hence, we have

(IA.I.3) $\quad \Sigma_1(t;\theta) = \sigma X(t)f^\theta_x(t, X(t)), \quad K_1(t,u;\theta) = f^\theta(t, e^uX(t^-)) - f^\theta(t, X(t^-)).$

Since $V_1(t;T,\theta) = f^\theta\big(t, X(t)\big)$ increases in the x argument for all $t \in [\tau_I, T]$, we have $\Sigma_1(t;\theta) > 0$ and $-\Sigma_1(t;\theta)\theta_W(t) \geq -\Sigma_1(t;\theta)\kappa$ for $\theta_W(t) \in [-\kappa, \kappa]$. Next, we consider the dt term involving the integral in (IA.I.2):

$$I^\theta = -\int_{\mathbb{R}} K_1(t,u;\theta)\theta_N(t,u)\nu(du) = I^\theta_+ + I^\theta_-,$$

where

(IA.I.4) $$I_+^\theta = -\int_{\mathbb{R}^+} K_1(t,u;\theta)\theta_N(t,u)\nu(du), \quad I_-^\theta = -\int_{\mathbb{R}^-} K_1(t,u;\theta)\theta_N(t,u)\nu(du).$$

Since $K_1(t,u;\theta) > (< 0)$ on $\mathbb{R}^+$ ($\mathbb{R}^-$) by the monotonicity of f^θ in x, we have $I_+^\theta \geq -\int_{\mathbb{R}^+} K_1(t,u;\theta)(1-e^{-M_1 u})\nu(du)$ and $I_-^\theta \geq 0$ for all t.

Denote $\theta^*(t) = \big(\theta_W^*(t), \theta_N^*(t,u)\big) = (\kappa, 1 - e^{-M_1 u}\mathbf{1}_{u\geq 0} - \mathbf{1}_{u<0})$. Given the above, the linear driver in (IA.I.2) is the smallest under $\theta^*(t)$ for all $\big(V_1(t;T,\theta), \Sigma_1(t;\theta), K_1(t,u;\theta)\big)$. Hence, by the comparison theorem for BSDE with jumps (Quenez and Sulem, 2013), $V_1(\tau_I;T,\theta^*)$ is the smallest.

Next, $V_1(\tau_I;T,\theta^*) = C_1 X(\tau_I)$, where $C_1 > 0$ is a constant, we can write $V_e(0;\tau_I,\theta^*)$ as

(IA.I.5) $$V_e(0;\tau_I,\theta^*) = \inf_{Q^\theta} \mathbb{E}_0^\theta\big[e^{-r\tau_I} C_1 X(\tau_I) - e^{-r\tau_I}\delta I\big].$$

Hence, under Q^θ, $\big(V_e(t;\tau_I,\theta), \Sigma_e(t;\theta), K_e(t,u;\theta)\big)$ is the solution to the following BSDE

$$-dV_e(t;\tau_I,\theta) = -\Sigma_e(t;\theta)dW^\theta(t) - \int_{\mathbb{R}} K_e(t,u;\theta)\tilde{N}^\theta(dt,du), \; t \in [0,\tau_I],$$
$$\text{and} \quad V_e(\tau_I;\tau_I,\theta) = C_1 X(\tau_I) - \delta I.$$

Again, by Girsanov's Theorem, we have $\big(V_e(t;\tau_I,\theta), \Sigma_e(t;\theta), K_e(t,u;\theta)\big)$ as the solution to the following BSDE under Q^0

(IA.I.6)

$$-dV_e(t;\tau_I,\theta) = \big(-\Sigma_e(t;\theta)\theta_W(t) - \int_{\mathbb{R}} K_e(t,u;\theta)\theta_N(t,u)\nu(du)\big)dt - \Sigma_e(t;\theta)dW(t)$$
$$- \int_{\mathbb{R}} K_e(t,u;\theta)\tilde{N}(dt,du),\ t \in [0,\tau_I],\ \text{and } V_e(\tau_I;\tau_I,\theta) = C_1X(\tau_I) - \delta I.$$

Similarly, by Itô formula, with $V_e(t;\tau_I,\theta) = f^\theta\big(t, X(t)\big)$, we have the same for $\Sigma_e(t;\theta)$ and $K_e(t,u;\theta)$ as in (IA.I.3).

Because (IA.I.5) suggests that $V_e(t;\tau_I,\theta) = f^\theta\big(t, X(t)\big)$ increases in the x argument for all $t \in [0,\tau_I]$, we have $\Sigma(t;\theta) > 0$ and $-\Sigma(t;\theta)\theta_W(t) \geq -\Sigma(t,\theta)\kappa$ for $\theta_W(t) \in [-\kappa,\kappa]$. Since $K(t,u;\theta) > (< 0)$ on $\mathbb{R}^+$ ($\mathbb{R}^-$) by the monotonicity of f^θ in x, we have $I^\theta_+ \geq -\int_{\mathbb{R}^+} K(t,u;\theta)(1-e^{-M_1u})\nu(du)$ and $I^\theta_- \geq 0$ for all t for I^θ_+ and I^θ_- defined in the same ways as (IA.I.4).

Denote $\theta^*(t) = \big(\theta^*_W(t), \theta^*_N(t,u)\big) = (\kappa, 1 - e^{-M_1u}\mathbf{1}_{u\geq 0} - \mathbf{1}_{u<0})$ for $t \in [0,\tau_I]$. Given the above, the linear driver in (IA.I.6) is the smallest under $\theta^*(t)$ for all $\big(V(t;\tau_I,\theta), \Sigma(t;\theta), K(t,u;\theta)\big)$. Hence, by the comparison theorem for BSDE with jumps, $V(0;\tau_I,\theta^*)$ is the smallest.

Taken together, $\theta^*(t) = \big(\theta^*_W(t), \theta^*_N(t,u)\big) = (\kappa, 1 - e^{-M_1u}\mathbf{1}_{u\geq 0} - \mathbf{1}_{u<0})$ is the minimum-expectation density generator for all $t \in [0,T]$. □

Proposition 2

Proof. Under Q^{θ^*}, $X(t)$ follows

$$dX(t)/X(t^-) = \mu^{\theta^*}dt + \sigma dW^{\theta^*}(t) + \int_{\mathbb{R}}(e^u - 1)\tilde{N}^{\theta^*}(dt, du),$$

where $\mu^{\theta^*} = \mu - \kappa\sigma - \lambda p/(\eta_1 - 1) + \lambda p\eta_1/\big((\eta_1^*)^2 - \eta_1^*\big)$.

By Itô formula, we have

$$\begin{aligned}
&X(t) = X(0)e^{Y(t)}, \text{ where} \\
&Y(t) = (\mu^{\theta^*} - \frac{1}{2}\sigma^2)t + \sigma W(t) + t\int_{\mathbb{R}}(u - e^u + 1)\nu^{\theta^*}(du) + \int_0^t\int_{\mathbb{R}} u\tilde{N}^{\theta^*}(ds, du).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{T\to\infty}\mathbb{E}_{\tau_I}^{\theta^*}\Big[\int_{\tau_I}^T e^{-rt}X(t)dt\Big] = \lim_{T\to\infty} e^{-r\tau_I}X(\tau_I)\mathbb{E}_{\tau_I}^{\theta^*}\Big[\int_{\tau_I}^T e^{-r(t-\tau_I)+Y(t-\tau_I)}dt\Big] \\
&= \lim_{T\to\infty} e^{-r\tau_I}X(\tau_I)\int_{\tau_I}^T \mathbb{E}_{\tau_I}^{\theta^*}\Big[e^{-r(t-\tau_I)+Y(t-\tau_I)}\Big]dt = \lim_{T\to\infty} e^{-r\tau_I}X(\tau_I)\int_{\tau_I}^T e^{-(r-\mu^{\theta^*})(t-\tau_I)}dt \\
&= \frac{e^{-r\tau_I}X(\tau_I)}{r - \mu^{\theta^*}}.
\end{aligned}$$

Then,

$$V_e(0; \tau_I^e, \theta^*) = \sup_{\tau_I}\ \mathbb{E}_0^{\theta^*}\big[e^{-r\tau_I}\big(A_0X(\tau_I) - I\big)\big], \quad A_0 = \frac{1-\phi}{r - \mu^{\theta^*}}.$$

The functional form inside the conditional expectation entails that the optimal stopping time is of the threshold type $\tau_I^e = \inf_t\{X(t) \geq X_I^e\}$. However, due to the "overshooting" problem caused by jumps, it is also possible that $X(\tau^e) > X_I^e$. Hence, we utilize the results from Kou and

Wang (2003) for the following

(IA.I.7) $$\mathbb{E}_0^{\theta^*}\left[e^{-r\tau_I}\right] = c_{1,0}\Big(\frac{X_I}{x}\Big)^{-\beta_1} + c_{2,0}\Big(\frac{X_I}{x}\Big)^{-\beta_2},$$

and

(IA.I.8) $$\mathbb{E}_0^{\theta^*}\left[e^{-r\tau_I}X(\tau_I)\right] = X_I\left[c_{1,1}\Big(\frac{X_I}{x}\Big)^{-\beta_1} + c_{2,1}\Big(\frac{X_I}{x}\Big)^{-\beta_2}\right],$$

where $\tau_I = \inf_t\{X(t) \geq X_I\}$ for some $X_I \geq X(0) = x$, and β_i and $c_{i,j}$ are given by (10) and (11). Hence,

$$V_e(0;\tau_I,\theta^*) = A_0X_I\left[c_{1,1}\Big(\frac{X_I}{x}\Big)^{-\beta_1} + c_{2,1}\Big(\frac{X_I}{x}\Big)^{-\beta_2}\right] - I\left[c_{1,0}\Big(\frac{X_I}{x}\Big)^{-\beta_1} + c_{2,0}\Big(\frac{X_I}{x}\Big)^{-\beta_2}\right]$$

Thus, we can utilize the "smooth pasting" principle, or Theorem 3.2 of Øksendal and Sulem (2019), that $V_e(0;\tau_I^e,\theta^*)$ as a function of $X(0) = x$ should be C^1 at X_I^e to be optimal to find the free boundary X_I^e. Because the following equation holds

$$\frac{\partial V_e}{\partial x} = B_1X_I^{1-\beta_1}x^{\beta_1-1} + B_2X_I^{1-\beta_2}x^{\beta_2-1} - B_3X_I^{-\beta_1}x^{\beta_1-1} - B_4X_I^{-\beta_2}x^{\beta_2-1},$$

with

$$B_1 = \beta_1c_{1,1}A_0, \quad B_2 = \beta_2c_{2,1}A_0, \quad B_3 = \beta_1c_{1,0}I, \quad \text{and} \quad B_4 = \beta_2c_{2,0}I,$$

we have

$$\frac{\partial V_e}{\partial x}\Big|_{x \uparrow X_I^e} = B_1 + B_2 - B_3/X_I^e - B_4/X_I^e. \tag{IA.I.9}$$

At X_I^e, (IA.I.9) satisfies

$$\frac{\partial V_e}{\partial x}\Big|_{x \downarrow X_I^e} = A_0.$$

Hence,

$$X_I^e = \frac{I}{A_0} \frac{\beta_1 \beta_2}{(\beta_1 - 1)(\beta_2 - 1)} \frac{\eta_1^* - 1}{\eta_1^*}.$$

□

Proposition 3

Proof. The proof is similar to the proof of Proposition 1. We apply the comparison theorem for BSDE with jumps. Since equity value increases in the x argument, the linear driver is the smallest under the $\theta^*(t)$ stated.

□

Proposition 4

Proof. Under Q^{θ^*}, $X(t)$ follows

$$dX(t)/X(t^-) = \mu^{\theta^*} dt + \sigma dW^{\theta^*}(t) + \int_{\mathbb{R}} (e^u - 1) \tilde{N}^{\theta^*}(dt, du),$$

where $\mu^{\theta^*} = \mu - \kappa\sigma - \lambda p/(\eta_1 - 1) + \lambda p \eta_1 / \big((\eta_1^*)^2 - \eta_1^*\big)$.

According to Dynkin's formula,

$$\mathbb{E}_{\tau_I}^{\theta^*}\Big[\int_{\tau_I}^{\tau_D} e^{-r(t-\tau_I)}(1-\phi)(X(t)-C)dt\Big] = (1-\phi)\Big[\frac{X(\tau_I)}{r-\mu^{\theta^*}}-\frac{C}{r}-\mathbb{E}_{\tau_I}^{\theta^*}\Big[e^{-r(\tau_D-\tau_I)}\Big(\frac{X(\tau_D)}{r-\mu^{\theta^*}}-\frac{C}{r}\Big)\Big]\Big].$$

Then,

$$\text{(IA.I.10)}\quad E(\tau_I;\tau_D^*,\theta^*,C) = \sup_{\tau_D\geq\tau_I}(1-\phi)\Big[\frac{X(\tau_I)}{r-\mu^{\theta^*}}-\frac{C}{r}+\mathbb{E}_{\tau_I}^{\theta^*}\Big[e^{-r(\tau_D-\tau_I)}\Big(\frac{C}{r}-\frac{X(\tau_D)}{r-\mu^{\theta^*}}\Big)\Big]\Big],$$

given that $Q^{\theta^*}\{\tau_D^*<\infty\}=1$.

The functional form inside the conditional expectation entails that the optimal stopping time is of the threshold type $\tau_D^* = \inf_t\{X(t)\leq X_D^*\}$. However, due to the "overshooting" problem caused by jumps, it is also possible that $X(\tau_D^*) < X_D^*$. Hence, we utilize the results from Kou and Wang (2003) for the following

$$\mathbb{E}_{\tau_I}^{\theta^*}\Big[e^{-r(\tau_D-\tau_I)}\Big] = d_{1,0}\Big(\frac{X_D}{x}\Big)^{\beta_3}+d_{2,0}\Big(\frac{X_D}{x}\Big)^{\beta_4},$$

and

$$\mathbb{E}_{\tau_I}^{\theta^*}\Big[e^{-r(\tau_D-\tau_I)}X(\tau_D)\Big] = X_D\Big[d_{1,1}\Big(\frac{X_D}{x}\Big)^{\beta_3}+d_{2,1}\Big(\frac{X_D}{x}\Big)^{\beta_4}\Big],$$

where $\tau_D = \inf_t\{X(t)\leq X_D\}$ for some $X_D\leq X(\tau_I)=x$ and $d_{i,j}$ and β_i are given by (14) and (10). Hence,

$$E(\tau_I;\tau_D,\theta^*,C) = (1-\phi)\Big[\frac{x}{r-\mu^{\theta^*}}-\frac{C}{r}\Big]+\frac{(1-\phi)C}{r}\Big[d_{1,0}\Big(\frac{X_D}{x}\Big)^{\beta_3}+d_{2,0}\Big(\frac{X_D}{x}\Big)^{\beta_4}\Big]$$
$$-\frac{(1-\phi)X_D}{r-\mu^{\theta^*}}\Big[d_{1,1}\Big(\frac{X_D}{x}\Big)^{\beta_3}+d_{2,1}\Big(\frac{X_D}{x}\Big)^{\beta_4}\Big].$$

Therefore, we can utilize the "smooth pasting" principle that $E(\tau_I; \tau_D^*, \theta^*, C)$ as a function of $X(\tau_I) = x$ should be C^1 at X_D^* to be optimal to find the free boundary X_D^*. Since

$$\frac{\partial E}{\partial x}\Big|_{x \downarrow X_D} = \frac{1-\phi}{r-\mu^{\theta^*}} - \frac{(1-\phi)C}{rX_D}(\beta_3 d_{1,0} + \beta_4 d_{2,0}) + \frac{1-\phi}{r-\mu^{\theta^*}}(\beta_3 d_{1,1} + \beta_4 d_{2,1}),$$

and $E(X_D^-) = 0$, we must have

$$\frac{1-\phi}{r-\mu^{\theta^*}} - \frac{(1-\phi)C}{rX_D^*}(\beta_3 d_{1,0} + \beta_4 d_{2,0}) + \frac{1-\phi}{r-\mu^{\theta^*}}(\beta_3 d_{1,1} + \beta_4 d_{2,1}) = 0.$$

Hence,

(IA.I.11)
$$X_D^* = \frac{C(r-\mu^{\theta^*})}{r}\frac{\beta_3 d_{1,0} + \beta_4 d_{2,0}}{1+\beta_3 d_{1,1} + \beta_4 d_{2,1}} = \frac{C(r-\mu^{\theta^*})}{r}\frac{\beta_3\beta_4(\eta_2^* + 1)}{(\beta_3+1)(\beta_4+1)\eta_2^*}.$$

□

Proposition 5

Proof. For the first part of the statement, the proof proceeds in the same way as the proof of Proposition 1, i.e., to find the minimum possible linear driver of the associated BSDEs. We begin with an arbitrary large horizon T. Since the horizon consists of two parts, $[\tau_I, \tau_D^*]$ and $[\tau_D^*, T]$, we examine them separately as in Proposition 1. Since in both horizons, debt value increases in the x argument, the stated $\theta^*(t)$ delivers the smallest linear driver.

Now we can evaluate $D(\tau_I; \tau_D^*, \theta^*, C)$ as

$$D(\tau_I; \tau_D^*, \theta^*, C) = \frac{C}{r}\big(1 - \mathbb{E}_{\tau_I}^{\theta^*}[e^{-r(\tau_D^* - \tau_I)}]\big) + A_1 \mathbb{E}_{\tau_I}^{\theta^*}[e^{-r(\tau_D^* - \tau_I)} X(\tau_D^*)].$$

Again, we use the results for the first passage probabilities to obtain

$$D(\tau_I;\tau_D^*,\theta^*,C)=\frac{C}{r}\Big[1-d_{1,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3}-d_{2,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big]+A_1X_D^*\Big[d_{1,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3}+d_{2,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big].$$

□

Proposition 6

Proof. Using the expressions for $E(\tau_I)$ and $D(\tau_I)$ in Propositions 4 and 5, we have

$$\begin{aligned}V_*(\tau_I;\tau_D^*,\theta^*,C)&=A_0X(\tau_I)+\frac{\phi C}{r}\Big[1-d_{1,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3}-d_{2,0}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big]\\&\quad-A_7X_D^*\Big[d_{1,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_3}+d_{2,1}\Big(\frac{X_D^*}{X(\tau_I)}\Big)^{\beta_4}\Big],\end{aligned}$$

where $A_7=\alpha A_0=\alpha(1-\phi)/(r-\mu^{\theta^*})$, and we have omitted I because it does not depend on C.

Then, the first order necessary condition for C^* is

$$\begin{aligned}&\frac{\phi}{r}-\frac{\phi}{r}d_{1,0}(1+\beta_3)\Big(\frac{\gamma C^*}{X(\tau_I)}\Big)^{\beta 3}-\frac{\phi}{r}d_{2,0}(1+\beta_4)\Big(\frac{\gamma C^*}{X(\tau_I)}\Big)^{\beta 4}\\&-A_7\gamma d_{1,1}(1+\beta_3)\Big(\frac{\gamma C^*}{X(\tau_I)}\Big)^{\beta_3}-A_7\gamma d_{1,2}(1+\beta_4)\Big(\frac{\gamma C^*}{X(\tau_I)}\Big)^{\beta_4}=0,\end{aligned}\tag{IA.I.12}$$

for all $X(\tau_I)$ values.

Denote $C^*=X(\tau_I)f(X(\tau_I))$. It suffices to show that $f(X(\tau_I))$ is a constant. Let $g(f(X(\tau_I)))$ be

$$\begin{aligned}g(f(X(\tau_I)))=&\frac{\phi}{r}-\frac{\phi}{r}d_{1,0}(1+\beta_3)\Big[\gamma f(X(\tau_I))\Big]^{\beta 3}-\frac{\phi}{r}d_{2,0}(1+\beta_4)\Big[\gamma f(X(\tau_I))\Big]^{\beta 4}\\&-A_7\gamma d_{1,1}(1+\beta_3)\Big[\gamma f(X(\tau_I))\Big]^{\beta_3}-A_7\gamma d_{1,2}(1+\beta_4)\Big[\gamma f(X(\tau_I))\Big]^{\beta_4}.\end{aligned}\tag{IA.I.13}$$

Then, the derivative of g with respect to the $f(X(\tau_I))$ argument is

(IA.I.14)

$$g'(f(X(\tau_I))) = -\frac{\phi}{r}d_{1,0}\beta_3(1+\beta_3)\gamma^{\beta_3}f(X(\tau_I))^{\beta_3-1} - \frac{\phi}{r}d_{2,0}\beta_4(1+\beta_4)\gamma^{\beta_4}f(X(\tau_I))^{\beta_4-1}$$
$$-A_7d_{1,1}\beta_3(1+\beta_3)\gamma^{1+\beta_3}f(X(\tau_I))^{\beta_3-1} - A_7d_{1,2}\beta_4(1+\beta_4)\gamma^{1+\beta_4}f(X(\tau_I))^{\beta_4-1}.$$

It is clear that $g'(f(X(\tau_I))) < 0$ because $f(X(\tau_I)) > 0$. Hence $g(f(X(\tau_I)))$ is strictly monotonic in $\mathbb{R}^+$, meaning that $g(f(X(\tau_I)))$ has a unique root in $\mathbb{R}^+$ if it exists. Clearly, $g(0) > 0$ and $g(\infty) = -\infty$, then the root exists.

Thus, we have shown that $f(X(\tau_I)) = \psi$ or $C^* = \psi X(\tau_I)$, where ψ satisfies

(IA.I.15)

$$\frac{\phi}{r} = \left(\frac{\phi}{r}d_{1,0}(1+\beta_3)\gamma^{\beta 3} + \frac{\alpha(1-\phi)d_{1,1}(1+\beta_3)\gamma^{1+\beta_3}}{r-\mu^{\theta^*}}\right)\psi^{\beta_3}$$
$$+\left(\frac{\phi}{r}d_{2,0}(1+\beta_4)\gamma^{\beta 4} + \frac{\alpha(1-\phi)d_{1,2}(1+\beta_4)\gamma^{1+\beta_4}}{r-\mu^{\theta^*}}\right)\psi^{\beta_4}.$$

Furthermore, the above analysis suggests that $V(\tau_I)$ is concave in C, sufficient for C^* to be value maximizing.

□

Proposition 7

Proof. Because the value function (19) is of the same type as Equation (IA.I.5) in Proposition 1, we can use the same analysis to show that $\theta^*(t) = (\theta^*_W(t), \theta^*_N(t,u)) = (\kappa, 1 - e^{-M_1 u}\mathbf{1}_{u\geq 0} - \mathbf{1}_{u<0})$ is the minimum-expectation density generator for all $t \in [0, \tau_I]$. Hence, we can write (19) as

$$V_*(0; \tau_I^*, \theta^*, C^*) = \sup_{\tau_I} \mathbb{E}_0^{\theta^*}\left[e^{-r\tau_I}\left(A_4X(\tau_I) - I\right)\right].$$

Clearly, the functional form in the conditional expectation operator indicates that τ_I^* is of the threshold type. We follow the same procedure as before by evaluating the value function under an arbitrary boundary first. Let $\tau_I = \inf_t\{X(t) \geq X_I\}$. Then,

$$\begin{aligned} V_*(0;\tau_I^*,\theta^*,C^*) &= \mathbb{E}_0^{\theta^*}\Big[e^{-r\tau_I}\big(A_4X(\tau_I)-I\big)\Big] \\ &= A_4X_I\Big[c_{1,1}\Big(\frac{X_I}{x}\Big)^{-\beta_1}+c_{2,1}\Big(\frac{X_I}{x}\Big)^{-\beta_2}\Big]-I\Big[c_{1,0}\Big(\frac{X_I}{x}\Big)^{-\beta_1}+c_{2,0}\Big(\frac{X_I}{x}\Big)^{-\beta_2}\Big], \end{aligned}$$

where $X(0)=x$, $c_{i,j}$, and β_i are the same as in Proposition 2.

The rest is the same as that in the proof of Proposition 2. We only replace A_0 with A_4.

□

IA.II. Detection-error probability

The key to the detection-error probability is the log of the Radon-Nikodym derivative $\zeta^{\theta^*}(t) = \ln(Z^{\theta^*}(t))$, where $Z^{\theta^*}(t) = dQ^{\theta^*}/dQ^0$. Under Q^0, $Z^\theta(t)$is

$$dZ^\theta(t)/Z^\theta(t^-) = -\theta_W(t)dW(t) - \int_{\mathbb{R}} \theta_N(t)\tilde{N}(dt,du).$$

Since $\theta^*(t) = (\theta_W^*(t),\theta_N^*(t)) = (\kappa, 1-e^{-M_1u}\mathbf{1}_{u\geq 0})$ for all t, we have

$$\text{(IA.II.1)} \qquad dZ^{\theta^*}(t)/Z^{\theta^*}(t-) = -\kappa dW(t) - \int_{\mathbb{R}^+}(1-e^{-M_1u})\tilde{N}(dt,du).$$

Hence, by Itô formula, we have

$$Z^{\theta^*}(t) = \exp\Big(-\kappa W(t) - \frac{1}{2}\kappa^2 t - \int_0^t \int_{\mathbb{R}^+} M_1 u \tilde{N}(ds, du) - t\lambda p \eta_1 \Big(\frac{M_1}{\eta_1^2} - \frac{1}{\eta_1} + \frac{1}{\eta_1 + M_1}\Big)\Big).$$

Thus,

$$\zeta^{\theta^*}(t) = -\Big(\frac{\kappa^2}{2} + \lambda p \eta_1 \Big(\frac{M_1}{\eta_1^2} - \frac{1}{\eta_1} + \frac{1}{\eta_1 + M_1}\Big)\Big)t - \kappa W(t) - \int_0^t \int_{\mathbb{R}^+} M_1 u \tilde{N}(ds, du).$$

A. Relative entropy growth

Given an alternative measure Q^θ and the reference measure Q^0, we can write the growth in entropy of Q^θ relative to Q^0 over the time interval $[t, t+\Delta t]$ as

$$G(t, t+\Delta t) = \mathbb{E}_t^{Q^\theta}\Big[\ln\Big(\frac{Z^\theta(t+\Delta t)}{Z_t^\theta}\Big)\Big], \quad \mathcal{R}(Z_t^\theta) = \lim_{\Delta t \to 0} \frac{G(t, t+\Delta t)}{\Delta t} \quad t \geq 0.$$

To calculate the above, it suffices to write $Z^\theta(t)$, especially $Z^{\theta^*}(t)$, under Q^θ or Q^{θ^*}. Using Girsanov's Theorem, we can write (IA.II.1) under Q^{θ^*} as

$$dZ^{\theta^*}(t)/Z^{\theta^*}(t^-) = \alpha_{\theta^*} dt - \kappa dW^{\theta^*}(t) - \int_{\mathbb{R}^+} (1 - e^{-M_1 u}) \tilde{N}^{\theta^*}(dt, du)$$

with

$$\alpha_{\theta^*} = \kappa^2 + \Big(\frac{1}{\eta_1} - \frac{2}{\eta_1 + M_1} + \frac{1}{\eta_1 + 2M_1}\Big)\lambda p \eta_1,$$

and $W^{\theta^*}(t)$ and $\tilde{N}^{\theta^*}(dt, du)$ being standard Brownian motion and compensated Poisson random measure under Q^{θ^*}. Here, $\tilde{N}^{\theta^*}(dt, du)$ has a Lévy measure $\nu^{\theta^*}(du)$ given by

$$\nu^{\theta^*}(du) = \lambda^* f_u^{\theta^*},$$

where

$$\lambda^* = \lambda(p\eta_1/(\eta_1 + M_1) + q), \quad f_u^{\theta^*} = p^*\eta_1^* e^{-\eta_1^* u} 1_{u \geq 0} + q^*\eta_2^* e^{\eta_2^* u} 1_{u<0}.$$

$$Z^{\theta^*}(t) = \exp\Big(\alpha_{\theta_*^*} t - \kappa W^{\theta^*}(t) - \int_0^t \int_{\mathbb{R}^+} M_1 u \tilde{N}^{\theta^*}(ds, du)\Big),$$

where $\alpha_{\theta_*^*}$ is a constant given by

$$\alpha_{\theta_*^*} = \frac{1}{2}\kappa^2 + \Big(\frac{1}{\eta_1} - \frac{2}{\eta_1 + M_1} + \frac{1}{\eta_1 + 2M_1}\Big)\lambda p \eta_1 - \Big(\frac{M_1}{(\eta_1^*)^2} - \frac{1}{\eta_1^*} + \frac{1}{\eta_1^* + M_1}\Big)\lambda^* p_L \eta_1^*.$$

Thus, we obtain

$$\ln\Big(\frac{Z^{\theta^*}(t + \Delta t)}{Z^{\theta^*}(t)}\Big) = \alpha_{\theta_*^*}\Delta t - \kappa\Big(W^{\theta^*}(t + \Delta t) - W^{\theta^*}(t)\Big) - \int_t^{t+\Delta t} \int_{\mathbb{R}^+} M_1 u \tilde{N}^{\theta^*}(ds, du),$$

$$G(t, t + \Delta t) = \mathbb{E}_t^{Q^{\theta^*}}\Big[\ln\Big(\frac{Z^{\theta^*}(t + \Delta t)}{Z^{\theta^*}(t)}\Big)\Big] = \alpha_{\theta_*^*}\Delta t,$$

and

$$\mathcal{R}(Z_t^{\theta^*}) = \lim_{\Delta t \to 0} \frac{G(t, t + \Delta t)}{\Delta t} = \alpha_{\theta_*^*}.$$

Thus, the constraint becomes

$$\mathcal{R}(Z_t^{\theta^*}) \leq h. \tag{IA.II.2}$$

Here, the value of h is to be determined by a specific detection-error probability. For example, we set h such that the detection-error probability is at least 0.1. Furthermore, we can decompose $h = h_W + h_N$, meaning that we restrict the robustness concerns for diffusion ambiguity and jump ambiguity explicitly. In this regard, κ^* and M_1^* depend on h_W and h_N, respectively.

B. Detection-error probabilities

The detection-error probability is defined as

$$\text{(IA.II.3)} \qquad \pi(t, n; h) = \frac{1}{2}\Big[Q^0\{\zeta^{\theta^*}(n) > 0|\mathcal{F}_t\} + Q^{\theta^*}\{\zeta^{\theta^*}(n) < 0|\mathcal{F}_t\}\Big], \quad t \geq 0, n = mT$$

where T is the number of years and m is the sampling frequency, and h denotes the upper bound for total robustness concern. Maenhout (2006) and Aït-Sahalia and Matthys (2019) provide a way to calculate this probability based on the characteristic functions of $\zeta^{\theta^*}(t)$ under Q^0 and Q^{θ^*}. That is

$$\pi(t, n; h) = \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \Big(\Re\Big[\frac{\hat{\zeta}^{\theta^*}_{\theta^*}(u, t, n)}{iu}\Big] - \Re\Big[\frac{\hat{\zeta}^{\theta^*}_0(u, t, n)}{iu}\Big]\Big)du,$$

where $i = \sqrt{-1}$ and $\Re(\cdot)$ denotes the real part of a complex number. Next, we can use the Lévy-Khintchine formula to calculate the two characteristic functions.

Nevertheless, explicitly, we can write

$$\begin{aligned}\hat{\zeta}^{\theta^*}_0(u, t, n) &= \mathbb{E}^{Q^0}_t[e^{iu\zeta^{\theta^*}(n)}] \\ &= \exp\Big((n-t)\Big(iu\alpha_0 - \frac{\kappa^2u^2}{2} + \lambda p\eta_1\Big(\frac{1}{iuM_1 + \eta_1} - \frac{1}{\eta_1} + \frac{iuM_1}{\eta_1^2}\Big)\Big)\Big),\end{aligned}$$

where

$$\alpha_0 = -\Big(\frac{\kappa^2}{2} + \lambda p \eta_1 \Big(\frac{M_1}{\eta_1^2} - \frac{1}{\eta_1} + \frac{1}{\eta_1 + M_1}\Big)\Big).$$

For $\hat{\zeta}_{\theta^*}^{\theta^*}(u,t,n)$ we have

$$\hat{\zeta}_{\theta^*}^{\theta^*}(u,t,n) = \mathbb{E}_t^{Q^{\theta^*}}[e^{iu\zeta^{\theta^*}(n)}] = \mathbb{E}_t^{Q^0}[e^{iu\zeta^{\theta^*}(n)}e^{\zeta^{\theta^*}(n)}] = \mathbb{E}_t^{Q^0}[e^{(iu+1)\zeta^{\theta^*}(n)}]$$

$$= \exp\Big((n-t)\Big((iu+1)\alpha_0 + \frac{(iu+1)^2\kappa^2}{2} + \Big(\frac{1}{(iu+1)M_1 + \eta_1} - \frac{1}{\eta_1} + \frac{(iu+1)M_1}{\eta_1^2}\Big)\lambda p \eta_1\Big)\Big)$$

IA.III. Parameter estimation

We use the full historical series of standardized DJX option contracts from the Ivy DB OptionMetrics volatility-surface dataset. Our sample spans from the inception of these options on October 6, 1997, through August 31, 2023, which is the latest available date in the database. For each option, we also collect the corresponding closing index level, continuous dividend yield, and the term structure of risk-free interest rates provided by Ivy DB.

The standardized DJX options in the Volatility_Surface file are interpolated contracts defined on a fixed grid of maturities (10 to 730 calendar days) and deltas (0.10 to 0.90 for calls, with symmetric negatives for puts). For each maturity-delta pair, OptionMetrics reports the implied strike, the corresponding option premium, and the interpolated implied volatility, which we use in our estimation.

OptionMetrics constructs these series by computing the option price each day as the midpoint of the best bid and best ask across all exchanges at 15:59 ET, synchronized with the underlying index close. Contracts with non-standard settlements, bid-ask midpoints below intrinsic value, or failed implied volatility convergence are excluded. In addition, low-vega contracts are dropped when estimating the volatility surface to improve stability.

In our analysis, we focus on options with more than 30 days to expiration and moneyness in the range $S/K \in (0.95, 1.05)$, as these contracts tend to be more liquid. Moreover, since our model is based on an infinite horizon, incorporating longer-dated options is more appropriate for estimation.

For trading date t, we obtain parameter estimates ($\{\hat{\sigma}, \hat{\lambda}, \hat{\eta_1}, \hat{\eta_2}, \hat{p}\}_t$) by minimizing the sum of squared errors between model-implied and market-observed put option prices:

$$\min_{\{\sigma,\lambda,\eta_1,\eta_2,p\}_t} \sum_{K,T} \left(P_t^{(O)}(K,T) - P_t(K,T|\sigma,\lambda,\eta_1,\eta_2,p) \right)^2,$$

where $P_t^{(O)}(K,T)$ denotes the market observed price for maturity T and the strike price K, and $P_t(K,T|\sigma,\lambda,\eta_1,\eta_2,p)$ denotes the model price as a function of the parameters for the same maturity and strike.

To compute the model-implied European option price, we employ the Fourier transform approach developed by Lewis (2001). Let $S(t)$ be the index dynamics under the risk-neutral measure Q following

$$dS(t)/S(t) = (r-d)dt + \sigma dW(t) + \int_R (e^u - 1)\tilde{N}(dt, du),$$

where d is the continuous dividend yield, and the other terms are the same as in the paper. We can also write $S(t) = S(0)e^{(r-d)t+Y(t)}$, such that $Y(t)$ is a Lévy process and $e^{Y(t)}$ is a martingale with

$Y(0) = 0$. Hence, $Y(t)$ follows

$$\begin{aligned} Y(t) &= -\frac{1}{2}\sigma^2 t + t\int_R (u - e^u + 1)\nu(du) + \sigma W(t) + \int_0^t \int_R u\tilde{N}(ds, du) \\ &= \alpha t + \sigma W(t) + \int_0^t \int_R u\tilde{N}(ds, du), \quad \alpha = -\frac{1}{2}\sigma^2 + \lambda(1 + \frac{p}{\eta_1} - \frac{q}{\eta_2} - \frac{p\eta_1}{\eta_1 - 1} - \frac{q\eta_2}{\eta_2 + 1}). \end{aligned}$$

Hence, the characteristic function of $Y(t)$ is

$$\varphi(v) := \mathbb{E}[e^{ivY(t)}] = e^{t\psi(v)}, \quad \psi(v) = -\frac{1}{2}\sigma^2 v^2 + i\alpha v + \int_R (e^{ivu} - 1 - ivu)\nu(du).$$

The integral is

$$\begin{aligned} \int_R (e^{ivu} - 1 - ivu)\nu(du) &= \int_{R^-} (e^{ivu} - 1 - ivu)\lambda q\eta_2 e^{\eta_2 u} du + \int_{R^+} (e^{ivu} - 1 - ivu)\lambda p\eta_1 e^{-\eta_1 u} du \\ &= \lambda q\eta_2 \Big(\frac{1}{\eta_2 + iv} - \frac{1}{\eta_2} + \frac{iv}{\eta_2^2} \Big) + \lambda p\eta_1 \Big(\frac{1}{\eta_1 - iv} - \frac{1}{\eta_1} - \frac{iv}{\eta_1^2} \Big) \end{aligned}$$

Hence, collectively

$$\psi(v) = -\frac{1}{2}\sigma^2 v^2 - iv\Big(\frac{1}{2}\sigma^2 + \lambda\Big(\frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1\Big)\Big) + \lambda\Big(\frac{p\eta_1}{\eta_1 - iv} + \frac{q\eta_2}{\eta_2 + iv} - 1\Big)$$

Lewis (2001) shows that the call option $C(K, T)$ price is

$$C(K, T) = S(0)e^{-dT} - \frac{1}{\pi}\sqrt{S(0)K}e^{-(r+d)T/2} \int_0^\infty \Re\Big[e^{iuk}\varphi\big(u - \frac{i}{2}\big)\frac{1}{u^2 + \frac{1}{4}}\Big] du,$$

where $k := \ln(S(0)/K) + (r - d)T$, and $\Re(\cdot)$ denotes the real part of a complex number. Put option price follows the put-call parity:

$$P(K,T) = C(K,T) + Ke^{-rT} - S(0)e^{-dT}.$$

We loop over all trading dates to obtain a time series for each parameter and report the time-series mean in Table IA.V.1, using both the call and put samples. The estimates are similar across the two cases.

IA.IV. Comparative Statics

We conduct comparative statics analyses to explore how the benefits of debt and the related optimal capital structure decisions respond to different model features. In the baseline analysis, we assume that drift ambiguity and jump ambiguity contribute equally to relative entropy growth. As a first exercise, we vary the share of jump (drift) ambiguity and examine its impact on the equilibrium outcome. As a second exercise, we study how the length of the finite sample used to calibrate ambiguity affects the equilibrium. In both analyses, all other parameters are held constant at their benchmark values (see Table 2).

Jump ambiguity share

Figure IA.V.1 shows how the jump ambiguity share affects the benefits of debt and the driving forces. In this analysis, the detection-error probability is fixed at 5% when varying the share of jump ambiguity, which determines h^* through the relative contributions of jump ambiguity (M_1) and drift ambiguity (κ) to relative entropy growth. In Panel (a), the *base* X_I^e/X_I^* reaches its peak at 1.056 when ambiguity stems entirely from jumps ($h_W/h = 0$), declines monotonically to 1.0453 at $h_W/h = 0.9$, and rises slightly thereafter. In contrast, the project value gain increases with the share of drift ambiguity, peaking at 13.50% when $h_W/h = 0.7$, before declining to 11.84% at $h_W/h = 1$. The AD price gain follows a similar pattern. These dynamics reflect the competing effects of β_3 and β_1, which govern the *base* and *power* respectively. Panel (d) reveals that β_3 declines from 0.824 to 0.598 at $h_W/h = 0.9$ before modestly rebounding, while in Panel (e) β_1 rises from 2.000 to a maximum of 2.832 at $h_W/h = 0.8$ and then falls.

To further dissect the role of ambiguity share, we examine how it influences the key determinants of β_1 and β_3 in Figure IA.V.2. Recall that $-\beta_3$ (β_1) corresponds to the largest negative (smallest positive) root of $G(\beta) - r = 0$, where $G(1) - r = \mu^{\theta^*} - r < 0$. Consequently, β_3 is primarily governed by μ^{θ^*}: a lower μ^{θ^*} moves $-\beta_3$ closer to zero. That is, ambiguity affects β_3 mainly through its impact on μ^{θ^*}. By contrast, the effects on β_1 are more complex: while a lower μ^{θ^*} pushes β_1 away from one, a decrease in M_1 pulls β_1 toward one, albeit through a different channel.

Panel (a) shows that μ^{θ^*} declines with the drift ambiguity share h_W/h, reaching a minimum at $h_W/h = 0.85$ and then increasing modestly. This pattern reflects the functional form:

$$\mu^{\theta^*} = \mu - \kappa\sigma - \frac{\lambda p}{\eta_1 - 1} + \frac{\eta_1}{(\eta_1 + M_1 - 1)(\eta_1 + M_1)},$$

where μ^{θ^*} declines linearly in κ and quadratically in M_1. Panels (b) and (c) show that κ rises at a decreasing rate, while M_1 declines at an accelerating rate as h_W/h increases. Accordingly, changes in κ dominate the behavior of μ^{θ^*} at lower values of h_W/h, whereas changes in M_1 dominate at higher values of h_W/h. Thus, comparing Panel (d) of Figure IA.V.1 and Panel (a) of Figure IA.V.2, we observe that the behavior of β_3 mirrors that of μ^{θ^*}. For β_1, the initial rise exhibited in Panel (e) of Figure IA.V.1 reflects the stronger upward pressure from the decline in μ^{θ^*}, which offsets the downward pull from the gradual fall in M_1, as evident from Figure IA.V.2. At higher values of h_W/h, the combined effects of rising μ^{θ^*} and falling M_1 lead to the sharp decline in β_1.

In addition, μ^{θ^*} is lower under pure drift ambiguity ($h_W/h = 1$) than under pure jump ambiguity ($h_W/h = 0$). This asymmetry reflects differences in the maximally allowed relative entropy growth h^* across the two scenarios when the detection-error probability is held fixed. Panel (d) of Figure IA.V.2 shows that h^* is larger when drift ambiguity dominates. Intuitively, rare jump events—with large but infrequent impacts—are easier to be filtered from EBIT dynamics given sufficient data, whereas drift is more difficult to estimate, as emphasized by Merton (1980).

Figure IA.V.3 presents the results on the effect of ambiguity share on optimal capital structure decisions. Overall, the results are consistent with Remark 3 that jump ambiguity has limited influence on the optimal default decision, and they suggest that the investment acceleration benefit of debt—originating from the net tax benefit of debt—responds differently depending on the source of ambiguity. The net tax benefit is primarily governed by the expected first-passage time to reach a level from above, which is longer when jump ambiguity is the dominant concern.

When ambiguity arises entirely from jumps ($h_W/h = 0$), distinguishing the worst-case measure from the reference measure becomes relatively easy, which limits the magnitude of relative entropy growth, given that the detection-error probability is fixed at 5%. It is worth noting that in the worst-case scenario dominated by jump ambiguity, the unconditional probability of positive jumps and the mean positive jump size are minimized. However, positive jumps still occur with sizable probability (as shown in Figure 2). Thus, unlike drift ambiguity, the impact of jump ambiguity on drift distortion is second-order. Due to the endogenous distortions to the worst-case parameters caused by the rise in the share of drift ambiguity (shown in Figure IA.V.2), the expected first-passage time to reach a level from above gradually shortens, leading to declines in the optimal default boundary. The optimal coupon, levered equity value, debt value, leverage and the net tax benefit of debt fall as well, while the AD price of default and bankruptcy costs rise.

Interestingly, when h_W/h is close to 0.9, the optimal default boundary, the optimal coupon, levered equity value, debt value, leverage and the net tax benefit of debt start to rise slightly, while the AD price of default and bankruptcy costs fall moderately. This nonlinear pattern is mainly driven by the nonlinear relation between the drift distortion μ^{θ^*} and β_3—the key parameter that reflects the expected first-passage time to a level from above—and the ambiguity share h_W/h, as shown in Figures IA.V.1 and IA.V.2.

Our quantitative analysis reveals a key result: the relevance of drift versus jump ambiguity is context dependent, differing between optimal financing of a growth option and of an asset in place. The ambiguity share directly influences the first passage time to reach a level from above or below. In general, optimally financing an asset in place involves only the first passage time to reach a level from above, i.e., default time. In contrast, optimally financing a growth option involves both the first passage time to reach a level from above and below. Hence, the ambiguity share that leads to the lowest optimal leverage does not necessarily leads to the smallest investment acceleration effect of debt. This distinction suggests that financing policies should be tailored to the nature of the project, as ambiguity may affect innovation investments differently from assets in place. Thus, our findings imply that the valuation of ambiguity hinges on the decision context—whether the firm is optimally financing a growth option or an asset in place.

Sample length

As discussed in Section B, a shorter sample available to the decision maker implies less capability of distinguishing the worst-case measure from the reference measure, leading to greater scope of ambiguity and a higher value of h^* for any fixed detection-error probability. Table IA.V.2 reports the results when varying the sample size (n) of the artificial EBIT data used to calibrate detection-error probabilities. Intuitively, a larger (smaller) sample provides agents with more (less) information about the underlying data-generating process, thereby reducing (increasing) the extent of ambiguity.

In this context, sample size also proxies for the novelty of innovation: larger (smaller) samples correspond to less (more) novel projects with more (less) available information, leading to lower (higher) ambiguity about future EBIT. Novelty can therefore be proxied by the scarcity of relevant historical precedents for the project's cash flows. In Bayesian learning models, posterior precision increases with the amount of relevant data, so fewer comparable historical observations imply greater uncertainty about profitability (Pástor and Veronesi, 2003), particularly for young firms or technologies. In accounting and innovation research, novelty is documented to create information frictions that reduce forecastability (Glaeser and Lang, 2024; Zhang, 2006). Accordingly, we interpret a shorter effective sample as indicating a more novel project whose EBIT profile is less likely to resemble historical EBIT and is more subject to ambiguity.

We consider two alternative cases in which the entrepreneur has access to either 60 or 15 years of quarterly data, corresponding to $n = 240$ or $n = 60$. The weights on the two types of ambiguity are assumed to be equal. The results show that greater ambiguity, which is driven by a shorter sample and more limited information, leads to larger relative gains in project value. Halving the sample size nearly doubles the level of total ambiguity, and the reverse holds when the sample size increases. The *base* declines only slightly across the three cases $N = 60 \times 4$,

$N = 30 \times 4$, and $N = 15 \times 4$, consistent with the small changes in β_3. By contrast, the sharp rise in β_1 accounts for the sizable increase in relative gains. Overall, these findings suggest that debt financing is especially valuable for novel innovation projects.

IA.V. Tables and figures

TABLE IA.V.1

Summary of the parameter estimates.

We utilize the historical series of standardized DJX option contracts from the Ivy DB OptionMetrics volatility surface dataset, covering the period from October 6, 1997, to August 31, 2023. $S/K \in (0.95, 1.05)$. $T > 30$.

	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{p}$	$\hat{\eta}_1$	$\hat{\eta}_2$
Call	0.118	1.938	0.118	8.210	8.446
Put	0.118	1.986	0.112	8.343	8.482

TABLE IA.V.2

Comparative statics: Sample length

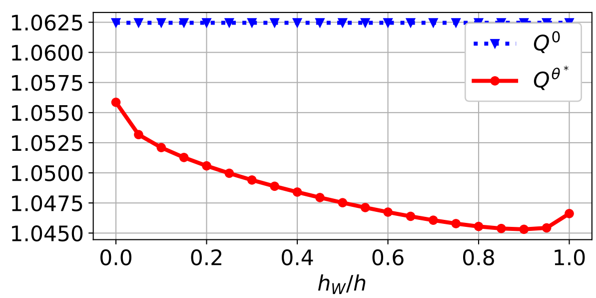
The quantities in the column Q^0 are computed using the baseline parameters (Table 2). Under the column Q^{θ^*}, quantities are computed at the maximum relative entropy growth h^*. The detection-error probability is fixed at 5%, using N quarters of data and $h_W/h = 0.5$.

	Q^0	Q^{θ^*}		
		$N = 60 \times 4$	$N = 30 \times 4$	$N = 15 \times 4$
h^*	0.000	0.026	0.049	0.093
κ	0.000	0.160	0.222	0.304
M_1	0.000	2.504	3.932	6.567
η_1^*	8.210	10.714	12.142	14.777
λ^*	1.986	1.931	1.910	1.882
p^*	0.118	0.093	0.083	0.069
μ^*	0.020	-0.013	-0.024	-0.039
X_I^e/X_I^*	1.062	1.051	1.048	1.044
$AD_I^*/AD_I^e - 1$ (%)	9.252	12.087	13.302	14.987
$V_I^*/V^e - 1$ (%)	9.252	12.087	13.302	14.987
β_1	1.461	2.295	2.690	3.267
β_3	0.978	0.716	0.643	0.565
X_D^*	0.272	0.222	0.206	0.189
AD_D	0.254	0.318	0.341	0.371
C^*	0.842	0.388	0.326	0.272
Equity	14.924	7.655	6.612	5.673
Debt	13.408	5.713	4.645	3.713
Leverage	0.473	0.427	0.413	0.396
Net Tax Benefit	1.666	0.648	0.511	0.393
Cost of Debt (%)	6.281	6.799	7.021	7.327

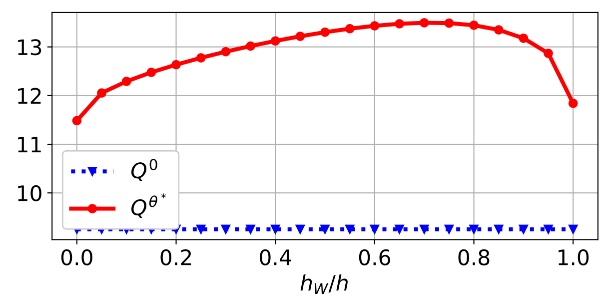
FIGURE IA.V.1

The value-enhancing effect of debt: ambiguity share

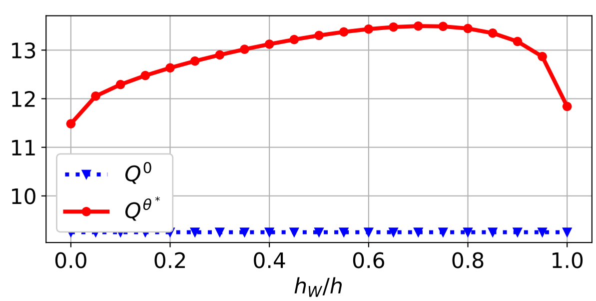
This figure plots X_I^e/X_I^*, $AD^*/AD^e - 1$, $V_*(0)/V_e(0) - 1$, β_3, and β_1 against h_W/h under the reference measure Q^0 (dashed blue triangle) and under the worst-case measure Q^{θ^*} (red solid dot) at the maximum ambiguity level h^*. The detection-error probability is fixed at 5% with $n = 120$ (quarters).



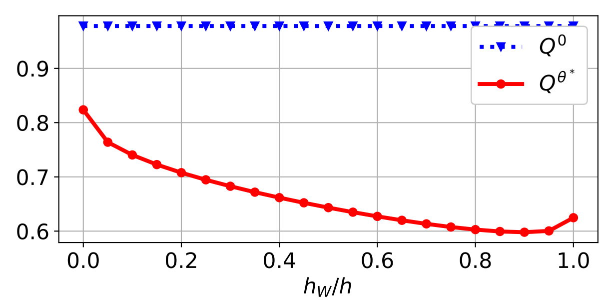
(a) X_I^e/X_I^*



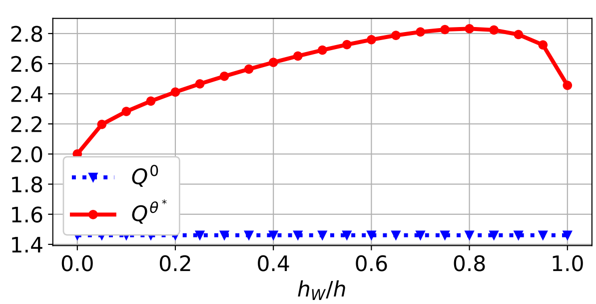
(b) $AD^*/AD^e - 1$ (%)



(c) $V_*(0)/V_e(0) - 1$ (%)



(d) β_3

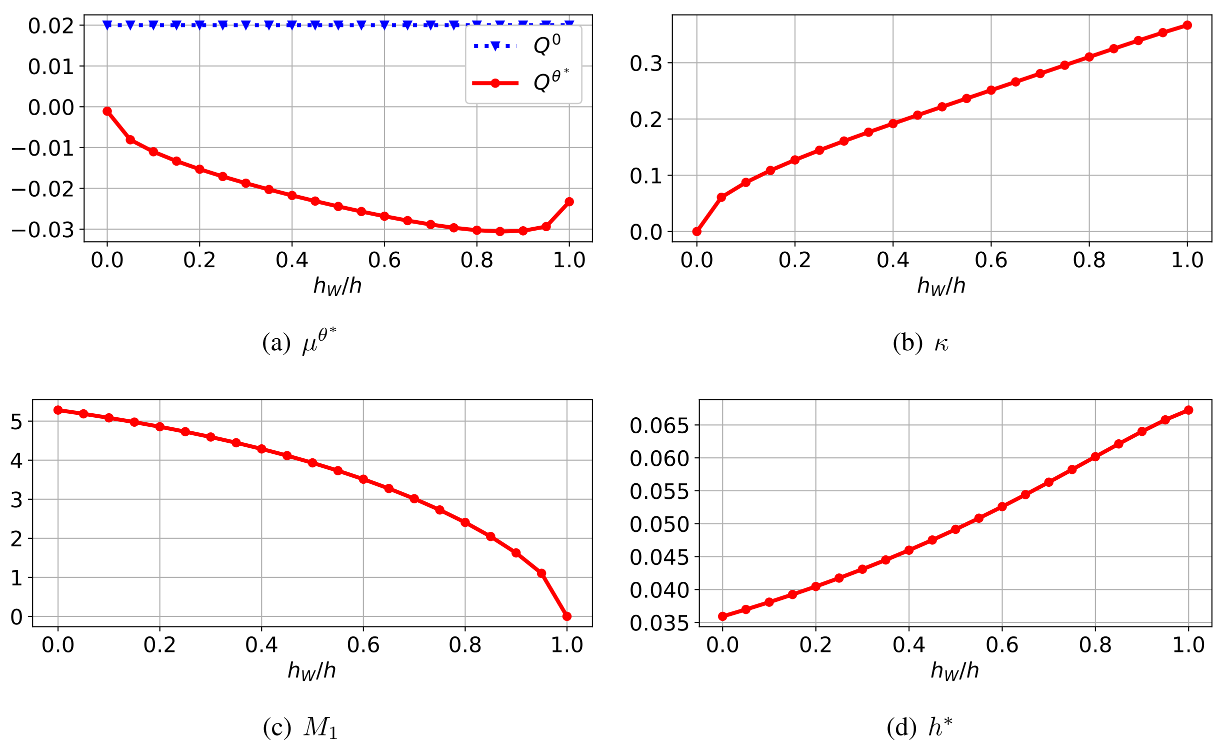


(e) β_1

FIGURE IA.V.2

Dissecting the ambiguity share effect

This figure plots μ^{θ^*}, κ, M_1, and h^* against h_W/h. The detection-error probability is fixed at 5% with $n = 120$ (quarters).



(a) μ^{θ^*}

(b) κ

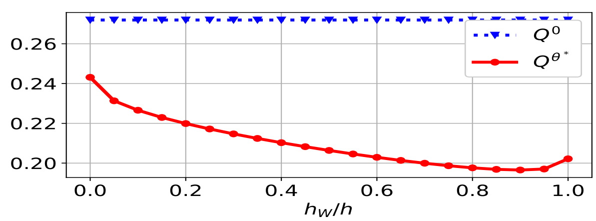
(c) M_1

(d) h^*

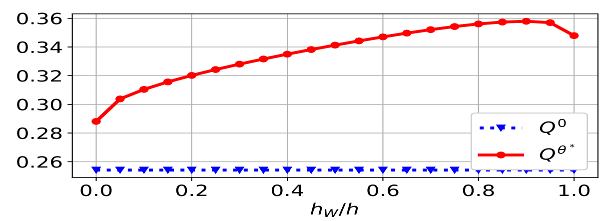
FIGURE IA.V.3

Optimal capital structure and ambiguity share

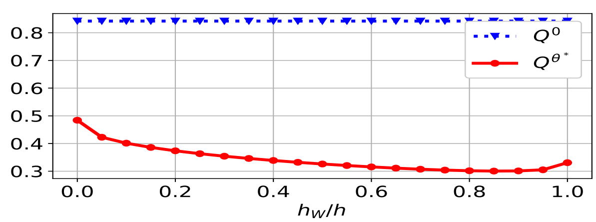
This figure plots the the scaled optimal default boundary X_D^*, the AD price of default, the scaled optimal coupon C^*, the scaled levered equity and debt, optimal leverage $D/(D+E)$, the scaled tax benefit, and the scaled bankruptcy cost against h_W/h. The scaled quantities are divided by $X(\tau_I^*)$. The upper bound of h is set such that the detection-error probability is 5% with $n = 120$ (quarters).



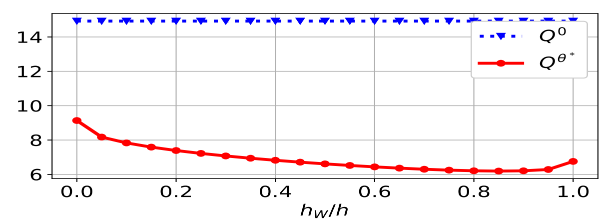
(a) X_D^*



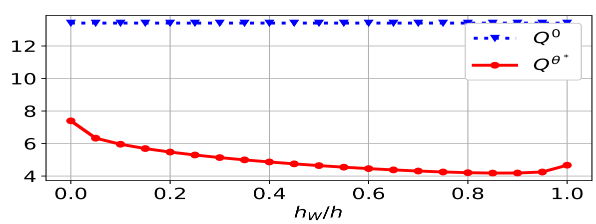
(b) AD Price (Default)



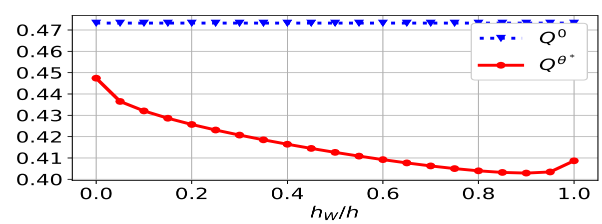
(c) C^*



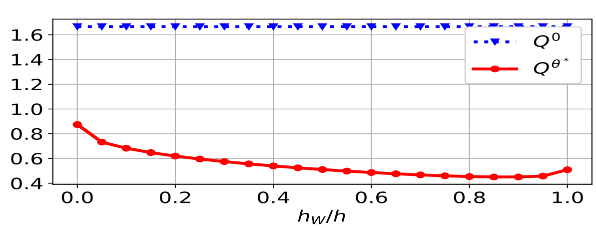
(d) Levered Equity



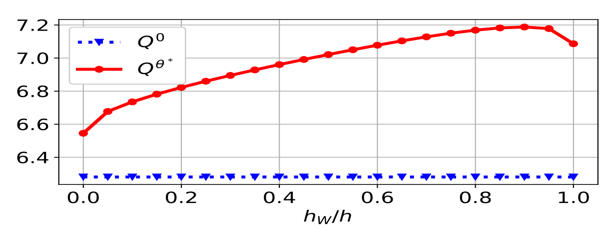
(e) Debt



(f) Leverage



(g) Net Tax Benefit



(h) Cost of Debt (%)

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